Reply to Florio and Shapiro

Abstract
Florio and Shapiro take issue with an argument in 'Hierarchies' for the conclusion that the set theoretic hierarchy is open-ended. Here we clarify and reinforce the argument in light of their concerns.

1.
We had two main aims in 'Hierarchies'. Our first aim was to make sense of Gödel's striking remark that Zermelo-Fraenkel set theory is 'nothing else but a natural generalization of the theory of types' (Gödel 1933, p. 46). Our second aim was to show: (a) that some of the same considerations that have motivated philosophers to embrace second-order languages can be used to motivate quantifiers of higher and higher logical type, and (b) that the resulting type-theoretic hierarchy ends up being isomorphic, in a way we make precise, with the hierarchy of sets.

In their rich and interesting response, Florio and Shapiro (hereafter F&S) do not take issue with the main thrust of 'Hierarchies'. They focus, instead, on a remark that we make towards the end of the paper. Here is their useful summary:

> Linnebo and Rayo claim that, in light of the close connection between set theory and type theory, we should accept that the set-theoretic (ontological) hierarchy is indefinitely extensible *given that* it is natural to hold that the suitably idealized type theoretic (ideological) hierarchy is similarly open-ended. (pp. 2-3)

It is this that will be our focus in what follows.

2.
Let us start with some definitions. *Absolutism*, as we will understand it here, is the view that there is a definite totality of all objects. *Set Theoretic Absolutism* is a special case of Absolutism which F&S are particularly interested in, namely the view that there is a definite totality of all sets. One way of denying Set Theoretic Absolutism—and therefore Absolutism—is by endorsing *Set Theoretic Open-Endedness*: the claim that any definite totality of sets can be used to define an even larger such totality. Set Theoretic Open-Endedness has a type-theoretic counterpart, which might be labelled *Type Theoretic Open-Endedness*, namely the claim that any definite totality of type-theoretic languages can be used to define an even larger such totality.

These definitions allow us to formulate two claims we defend in 'Hierarchies':

Claim 1
The absolutist has good reason to accept Type-Theoretic Open-Endedness.

Claim 2
Claim 1 gives the absolutist some reason to abandon Absolutism.

Our argument for Claim 1 is based on the observation that Type-Theoretic Open-Endedness follows from Absolutism, in the presence of the following two principles:

T1
Let L be a type-theoretic language L. Then it is possible to characterize a generalized semantics for L (that is, a theory of all possible interpretations for L).

T2
Let D be a definite totality of languages. Then there is a union language, which pools together the expressive resources of each language in D. (Cf. p. 294 of 'Hierarchies')

[Proof sketch: We show in the paper that Absolutism entails that it is impossible to characterize a generalized semantics for a type-theoretic language L in a language of the same logical type of L. It is, however, possible to characterize a generalized semantics for L in a language L+ with quantifiers of strictly higher logical type. So T1 entails that, for any type-theoretic language L, there is a language L+ with quantifiers of strictly higher logical type. It is then straightforward to show Type-Theoretic Open-Endedness. For let D be a definite totality of type-theoretic languages. By T2, there is a union language L that pools together the expressive resources of every language in D. But, by the argument above, there is a language L+ with quantifiers of strictly higher logical type, and such a language cannot be in D.]

Our argument for Claim 2 is far more tentative. It is based on the idea that one of the main attractions of Absolutism is its avoidance of Set Theoretic Open-Endedness. If Claim 1 is correct, however, the absolutist has good reason to accept Type Theoretic Open-Endedness. But it is not clear that Set Theoretic Open-Endedness is any more problematic than its type-theoretic counterpart. So it is not clear that avoidance of Set-Theoretic Open-Endedness constitutes a real benefit for the absolutist.

3.
F&S observe that our principles T1 and T2 have set-theoretic analogues, which were set forth by Georg Cantor in 1883:

O1.
For any ordinal, there is a successor ordinal.

O2.
For any definite totality of ordinals, there is an ordinal which is its least upper bound.

And, as Cantor showed, these two principles entail that the collection of ordinals is open-ended.

[Proof sketch. Assume the collection On of all ordinals is definite. Then, by O2, it has a least upper bound a. And, by O1, a has a successor a+1, which is larger than any element of On. Contradiction. (See Cantor 1883, and Shapiro 2003.)]

In their reply, F&S suggest that anyone who accepts principles T1 and T2 should also accept principles O1 and O2. But O1 and O2 already entail that Absolutism is false. So there is not much to be gained by using T1 and T2 to argue against Absolutism via Claims 1 and 2, as we do in 'Hierarchies'. As F&S put it, 'it would be surprising if [T1 and T2] could undermine absolutism about sets in a way that [O1 and O2] do not' (p. 7).

We agree with F&S that the two arguments against Absolutism are structurally similar and are grateful for this astute observation. But what, exactly, follows from the structural similarity between Cantor's argument and ours?

Generally speaking, a structural similarity between two arguments is enough to guarantee that one argument is valid just in case the other is. But soundness is another matter. Clearly, arguments with a shared structure need not be equally convincing. Is one of the above arguments more convincing than the other?

Speaking for ourselves, we find both arguments quite convincing (when properly developed), and convincing to roughly the same extent. (This is part of the 'liberalism' that we embrace in the final section of 'Hierarchies'.) The point we wished to stress in our article, however, is that *our opponent*—an absolutist who denies the existence of open-ended hierarchies of any sort—should have a harder time resisting our ideological argument than Cantor's ontological one.

This is because an absolutist with sufficiently strong realist leanings is likely to see an asymmetry between the ontological hierarchy of sets and the ideological hierarchy of types. On the ontological side, she is likely to think that a set-theoretic term might be coherent and nonetheless fail to refer. For, in the ontological arena, there can be brute (and inexplicable) existence failures: coherence is no guarantee of existence. In the ideological arena, on the other hand, she might be more amenable to the thought that, in certain cases, coherence *is* sufficient for existence. For she might think that the intelligibility of a higher-level quantifier is enough to give it a place in our ideological hierarchy of types.

It is presumably because of this asymmetry that Boolos's arguments for plural quantification enjoy widespread support among absolutists. The absolutist can agree with Boolos's non-liberal views about ontology, and deny that there is any kind of set-like object with all sets as members, while taking a liberal view about ideology and embracing plural variables, each of which can refer, collectively, to absolutely all sets.

The point we were trying to make towards the end of 'Hierarchies' is that this asymmetry between the absolutist's attitudes towards ontology and ideology can be exploited to make progress in the debate about Absolutism.

4.

In the final section of their response, F&S explore some ways of resisting our argument for type-theoretic open-endedness. They open with the following claim:

> since the levels in the ideological hierarchy are indexed by ordinals, the process of expanding one's expressive resources by moving to higher types is open-ended only to the extent that one's system of indices—the ordinals in this case—are open-ended. How could a definite stock of indices give rise to an open-ended hierarchy of languages?

It is worth noting, however, that there is no reason to think that the levels of the ideological hierarchy need to be indexed by ordinals. All we need is some system of 'labels' to attach to the expressions of a language in order to indicate their level. In fact, these 'labels' don't even need to be individual objects. Nothing prevents us from using pluralities—or super-pluralities, or super-duper-pluralities, or beyond—to serve this function. Doing so would enable us to describe larger and larger fragments of the ideological hierarchy. And the more of the hierarchy that can be described, the greater our resources for labeling additional levels of the hierarchy.

It goes without saying that it would be difficult to find a practical notation for such systems of indices. But that is not a threat to the intelligibility of the expressive resources that the indices might be used to characterize. Notice also that, regardless of one's choice of indices, principles T1 and T2 must be formulated using a language of strictly higher type than each of the languages under discussion. So it would be a mistake to think that T1 and T2 can be used to bootstrap one's way to an understanding of type-theoretical vocabulary that one did not previously understand. But, as before, this is no threat to the intelligibility of the relevant vocabulary. (It is important to keep in mind that T1 and T2 are set forth as principles constraining the structure of the hierarchy of types, not as explications of type-theoretic vocabulary. The best way to acquire an understanding of a language of very high type is presumably to become immersed in its use, with the aid of a suitable deductive system.)

5.

F&S point out that our principle T2 (like Cantor's O2) invokes 'an unanalysed notion of 'definite''. It is worth pointing out, however, that we are not the only ones to use such a notion. The notion of definiteness is *presupposed* by the debate between absolutists and proponents of open-endedness, since both Absolutism and Open-Endedness are explicitly defined in terms of definiteness. Notice, moreover, that it would be inappropriate to treat 'definite totality' as meaning 'set-sized totality', since doing so would trivialize the debate by reducing Set Theoretic Open-Endedness to the truism that for any set of sets, there is a larger set.

Still, there is no denying that the notion of definiteness deserves greater explanation. We end with a brief remark on this difficult topic.

F&S follow Cantor in describing definite totalities as 'changeless' (p. 7). On our view, this is potentially misleading, even if it is meant metaphorically. To describe the hierarchy of sets as open-ended is *not* to suggest that the feature of reality that set-theory aims to describe is, in some sense, subject to change. What is subject to change is our *concept* of set. More precisely, our concept of set is tied up with the sorts of linguistic resources that we have available for talking about sets. The result is that as our resources expand, so does our concept of set. Accordingly, if there turns out to be no final answer to the question of what sorts of resources might be used to describe the world, there will also be no final answer to the question of how far our concept of set might develop. It is in this sense that our concept of set might be said to be 'indefinitely extensible'.

An absolutist might complain that even if this is right, and there is no final answer to the question of what resources one might use to describe the world, there is certainly a final answer to the question of how many objects there are. So there is a hard limit to how many objects one is in a positions to count as sets: there are at most as many sets as there are objects.[1] On our view, however, the absolutist is wrong to suppose that there is a final answer to the question of how many objects there are. Following Frege, we think that a single feature of reality can get carved up into objects in different ways when it is described using different linguistic resources. The more inclusive the notion of set we use to describe the (unchanging) feature of reality that constitutes the subject-matter of set-theory, the more objects this single feature of reality will get carved up into. It is in this sense that the ontological hierarchy of sets and the ontological hierarchy of types might be thought of as different perspectives on the same subject-matter, as we suggested in 'Hierarchies'.[2,3]

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[1] See, for instance, McGee 1997.

[2] For further philosophical discussion, see chapters 1 and 3 of Rayo forthcoming; for a technical implementation of the idea underlying Set-Theoretic Open Endedness, see Linnebo forthcoming.

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