On the open-endedness of logical space

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January 8, 2020

Modal logicism is the view that a metaphysical possibility is just a non-absurd way for the world to be. I argue that modal logicists will see metaphysical possibility as "open ended": any given possibilities can be used to characterize further possibilities. I then develop a formal framework for modal languages that is a good fit for the modal logicist and show that it delivers some attractive results.

1 Possibility and Absurdity

A *possibility*, as I will understand it here, is a way for the world to be. More specifically: it is a way for the world to be that is free from absurdity.[1]

When does a way for the world to be cross the line into absurdity? Because we don't have a notion of absurdity that is theoretically neutral, this a question on which reasonable people might disagree. It is, however, natural to suppose that a contradictory state of affairs would be absurd. If this is right, then a way for the world to be on which there are elephants and there are no elephants involves an absurdity. It is also natural to suppose that it would be absurd for something to fail to be identical to itself. If this is right, then a way for the world to be on which Heseprus (i.e. Venus) fails to be identical with Phosphorus (i.e. Venus) would involve an absurdity. On the other hand, is natural to suppose

[1]Formally speaking, talk of possibilities is to be cashed out using propositional variables. For instance, "there is a possibility according to which there are elephants" should be cashed out as:

$$\exists p\,[(p \neq \bot) \wedge (p \equiv \exists x\ \text{Elephant}(x))]$$

where "p" is a propositional variable, "$\bot$" refers to the absurd proposition, and "$\equiv$" expresses the propositional analogue of identity. (See Appendix A for details.)

that there is no absurdity in God's existing and no absurdity in God's failing to exist. So, on the notion of possibility that will be our present concern, there is a possibility according to which God exists and a possibility according to which God fails to exist.

But, as I said, reasonable people might disagree. Although it is natural to suppose that a contradictory state of affairs would be absurd, a dialetheist might disagree (Priest 1998). Although it is natural to suppose that it would be absurd for Hesperus to fail to be identical with Phosphorus, a friend of relative identity might disagree (Geach 1967). Although it is natural to suppose that there is no absurdity in God's failing to exist, a friend of the Ontological Argument might disagree (Anselm 1077–78).

Absurdity is not the same as nonsensicality. Whereas absurdity is the result of *over*-specifying a way for the world to be; nonsensicality is the result of *mis*-specifying a way for the world to be. When I consider a way for the world to be according to which Hesperus exists, I successfully specify a possibility; if I go on to add that Phosphorus fails to exist, I over-specify and end up with an absurd way for the world to be, rather than a possibility. But I have not lapsed into nonsense. On the other hand, if I try to specify a way for the world to be on which it is true of my sister—I actually have no sisters—that "she" is a plumber, I won't go wrong by over-specifying; rather, I will have failed to specify a way for the world to be altogether: I will have lapsed in to nonsense.

In adopting a particular conception of absurdity, one incurs significant theoretical commitments. Consider, for example, the debate between classical logicians and their intuitionist rivals. On one way of construing it, the debate is partly about the limits of absurdity: the classicist thinks it would be absurd for it to be the case that $\neg\neg p$ without its thereby being the case that p; the intuitionist disagrees. Because the intuitionist has a less expansive conception of absurdity than the classicist, she has additional theoretical resources to work with. Significantly, she is in a position to resist inferences from $\neg\neg p$ to p, which might be useful in, e.g. addressing certain paradoxes. But these additional theoretical resources come at a cost, since the intuitionist must make sense of scenarios at which $\neg\neg p$ is verified without p and is therefore confronted with potentially awkward question about when such scenarios obtain and why. The classicist, in contrast, is not committed to answering such questions. But she pays for the privilege by having fewer theoretical resources to work with.

Generally speaking, the fewer scenarios one counts as absurd, the greater the range of theoretical resources one will have at one's disposal, but the greater the range of open questions one will be confronted with. This tradeoff imposes an independent constraint on the notion of absurdity. For rather than deciding where to place the limits of absurdity on the basis of raw intuition, one can think of the decision as a cost-benefit analysis. On the one hand, one's conception of absurdity should be narrow enough to make room for a fruitful body of theory; on the other, one's conception of absurdity should be expansive enough to minimize the range of fruitless questions one is committed to treating as legitimate. These two constraints run in opposite directions. So one should aim to adopt a conception of absurdity that strikes a good balance between the two.[2] There is nothing wrong about using intuition to develop a preliminary conception of absurdity. But an unintuitive conception of absurdity can earn its keep by making way for a sufficiently fruitful body of theory.

There is no guarantee that disputes about the limits of absurdity can be settled by appeal to a cost-benefit analysis, since different theorists are likely to disagree about what counts as "fruitful". But an individual theorist might use a cost-benefit analysis to decide which conception of absurdity to adopt herself.

2 Modal Logicism

Modal logicism, as I will understand it here, is the view that a metaphysical possibility is just a possibility, in the sense of §1. In other words: the dividing line between metaphysical possibility and impossibility is just the dividing line between the coherent and the absurd.[3]

As a contrast to modal logicism, consider the conception of metaphysical possibility developed in (Lewis 1986*b*). Simplifying a bit, Lewis postulates a class of "natural" properties, which are assumed to be intrinsic, and claims that a metaphysical possibility is a distribution of pointwise instantiations of natural properties across an admissible spacetime structure. He also accepts a Principle of Recombination "according to which patching together parts of different possible worlds yields another possible world" (§1.8). But in order to keep his Principle of Recombination from generating a contradiction,[4] he is forced to claim that only "sufficiently small" spacetime structures are admissible: "Among the mathematical structures that might be offered as isomorphs of possible spacetimes, some would be admitted, and others would be rejected as oversized."

[2]For a more detailed discussion of these issues, see (Rayo 2013).

[3]I use "coherent" as the contradictory of "absurd". In other words: a way for the world to be is coherent just in case it does not lapse into absurdity.

[4]The paradox is due to Forrest & Armstrong 1984.

(§2.2). Lewis does not advocate for a specific dividing line between acceptable and oversized spacetimes; he just insists that such a dividing line must exist somewhere. Offhand, however, there is nothing *absurd* about a spacetime that is well-defined mathematically but gets ruled out to save Lewis's Principle of Recombination. So there would appear to be possibilities that Lewis is not in a position to count as metaphysically possible.

Another example of a non-logicist conception of metaphysical possibility is Williamson's (2013) *necessitism*, which I shall understand as the conjunction of two claims:

Necessitist Axiom As a matter of necessity, everything exists necessarily.[5]

Absolutism There is sense to be made of talk of absolutely general quantification: quantification over absolutely everything there is.

By Absolutism, there is a definite answer to the question of how many individuals there are. Regardless of what that answer turns out to be, there is no obvious absurdity in an "oversized" way for the world to be, on which there are even more individuals than that. But by the Necessitist Axiom, it is metaphysically impossible for there to be more individuals than there, in fact, are. So there would appear to be possibilities that the necessitist is not in a position to count as metaphysically possible.

Both the Lewisan and the nececissit could challenge my conclusions by adopting sufficiently revisionist notions of absurdity. In particular they could claim that an "oversized" spacetime, or an "oversized" collection of individuals, is not just metaphysically objectionable but absurd. It is important to be clear, however, that such maneuvering would come at a cost. For in setting the limits of absurdity, one is not merely taking a stand on how the world is: one is setting limits on admissible inquiry. In rejecting as absurd the hypothesis of an "oversized" spacetime, or an "oversized" collection of individuals, one is not just committed to thinking that the hypothesis is false. One is committed to thinking that the hypothesis is *beyond consideration*: it is a mistake to even pose the question of whether the hypothesis is true, regardless of one's views about the answer. Perhaps there are good reasons for so limiting the range of admissible inquiry, but it is not clear to me that defending a preferred conception of metaphysical possibility is good enough.

[5]Here and throughout I use "exists" as a stylistic variant of "is identical to something".

3 Open-Endedness

We shall say, of some possibilities, that they constitute a *partition of logical space* if exactly one of them will be realized however the world is.

A partition of logical space needn't be *maximally specific*: it needn't be such that each of its members fully determines how the world is. Consider, for instance, the possibility that there be elephants and the possibility that there be none. Together they constitute a partition of logical space, since exactly one of them will be realized however the world is. But they do not constitute a maximally specific partition, since neither settles the question of whether Socrates wore a hat.

I claim that there is no such thing as a maximally specific partition of logical space. (For short: logical space is open-ended.) More precisely, I claim that the following is true, on natural assumptions about absurdity:

Open-Endedness Let some possibilities, ww, constitute a partition of logical space. Then ww can be be used to show that there are possibilities not entailed by any possibility among ww.[6]

There are many different ways of arguing for Open-Endedness,[7] but this is the one I like best:

[6]As is customary, I use a double variable as a plural variable, which refers collectively to one ore more objects, relative to a variable assignment (Boolos 1984, 1985); "$z \prec xx$" is read "z is one of xx" and "$yy \preceq xx$" is read "yy are among xx".

[7]Here is another: for each subplurality vv of ww, consider a way for the world to be π_{vv} according to which I prefer only vv. A variant of this argument, known as Kaplan's Paradox, is discussed in Davies 1981, Forrest & Armstrong 1984, Kaplan 1995, Nolan 1996, Fine 2003, Uzquiano 2015*a*, Fritz 2017. (For an objection to the view that the π_{vv} are genuine possibilities, see Lewis 1986*a*, §2.3.) I learned of a different argument from Gabriel Uzquiano: let rr be the possibilities π among ww that satisfy the following condition: $\forall vv \preceq ww$ ((at π, I prefer vv) $\rightarrow \pi \not\prec vv$); let ρ be a way for the world to be such that: $\forall vv \preceq ww$ ((at ρ, I prefer vv) $\rightarrow vv = rr$). Whether or not you think ρ is metaphysically possible, there is no apparent absurdity in ρ. And, on pain of contradiction, ρ cannot be entailed by a possibility among ww. (For related results, see Bacon & Uzquiano forthcoming.) It is worth noting that Uzquiano's argument makes no assumptions about the structure of propositions. In contrast, the Russell-Myhill Paradox assumes:

$$\forall x^{\langle\langle\rangle\rangle} \forall y^{\langle\langle\rangle\rangle} \forall p^{\langle\rangle} \forall q^{\langle\rangle} (x(p) \equiv y(q) \rightarrow x \equiv y \wedge p \equiv q.)$$

where the first two occurrences of "$\equiv$" express the analogue of identity for type $\langle\rangle$ and the third expresses the analogue of identity for type $\langle\langle\rangle\rangle$. (See Dorr 2017.)

How many island universes are there?

An island universe is a spatiotemporal manifold with no causal or spatiotemporal connections to any other part of the world. It is natural to assume that the existence of some island universes does not to impose any restrictions on the existence of other island universes. If so, any well-defined and pertinent answer to "How many island universes are there?" should count as non-absurd.

But the range of well-defined and pertinent answers to our question is open-ended: if you give me some individuals xx, I can use xx to show that there are *more* pertinent and well-defined answers to our question than there are xx.[8]

This applies, in particular, to the possibilities ww. So I can use ww to show that there are more such answers to our question than there are ww. Since each of these answers is non-absurd, it can be used to characterize a possibility: the possibility that there be just as many island universes as the answer claims. And since the answers are pairwise incompatible, the corresponding possibilities are also pairwise incompatible. So they cannot all be entailed by

[8]Here's one way to do it. By (a plural version of) the Axiom of Choice, there is a well-ordering $<$ of xx, which can be used to characterize the following function f, which assigns a proposition to each individual among xx. (Formally, f is a higher-order variable of type $\langle e \rangle$; see Appendix A for details.)

$$f(x) = \begin{cases} \left[\text{there are as many island universes as there are natural numbers}\right], & \text{if } x \text{ is } <\text{-smallest} \\ \left[\text{there are exactly as many island universes as there are subpluralities of island universes according to } f(y)\right], & \text{if } x \text{ has } y \text{ as its immediate } <\text{-predecessor} \\ \left[\text{there are at least as many island universes as there are island universes according to } f(y) \text{ for each } y < x\text{, but there are no more than that}\right], & \text{if } x \text{ has no immediate } <\text{-predecessor} \end{cases}$$

By (a plural generalization of) Cantor's Theorem, $f(x)$ and $f(y)$ are incompatible whenever $x \neq y$. So we have shown that there are at least as many pairwise incompatible answers to our question as there are xx. To show that there are *more* such answers, it suffices to move the argument up one level in the higher-order hierarchy, and define (the analogue) of f over subpluralities of xx.

possibilities among ww.

One can resist this argument by adopting a sufficiently revisionist notion of absurdity. But, as I noted in §2, revisionist conceptions of absurdity can come at a cost.

One can also accept the argument as stated, but resist the more ambitious conclusion that *metaphysical possibility* is open-ended. One might argue, in particular, that some of the possibilities involved in the argument flout a requirement of metaphysical possibility that is stricter than non-absurdity. Lewis might do so by insisting that spacetime not be "oversized". A necessitist might do so by insisting that the world's inhabitants not constitute an "oversized" collection of individuals.

4 Making Sense of Open-Endedness

In arguing that logical space is open-ended, I am not advocating the view that there is something "incomplete" or "open-ended" about the way the world is. On the contrary: I work on the assumption that there is a definite fact of the matter about how the world is. I claim not that the *world* is open-ended but that there is no such thing as an exhaustive collection of resources for *characterizing* the way the world is.

Here is the intuitive picture. To characterize the world is to set forth some distinctions between ways for the world to be and to coherently take sides with respect to each.[9] (For example, one can characterize the world by distinguishing between there being elephants and not, and taking sides in favor of the former.) But to set forth a distinction is just to divide logical space into distinct possibilities. (In our example, one divides logical space into the possibility that there be

[9]I restrict attention to *exclusive* distinctions: distinctions between the world's being so-and-so and its not being so-and-so. In addition, I assume that however the world happens to be, it will be both coherent and complete. More precisely:

Coherence However the world happens to be, it it will not involve an absurdity.

Completeness However the world happens to be, it will take sides with respect to every (exclusive) distinction.

Say that the possibilities *generated* by distinctions $\delta\delta$ are the ways for the world to be that result from taking sides with respect to every distinction among $\delta\delta$ while avoiding absurdity. Then our assumptions allow us to verify the following: whenever $\delta\delta$ are distinctions, the possibilities generated by $\delta\delta$ will always constitute a partition of logical space.

elephants and the possibility that there be none.) Accordingly, to characterize the world is just to divide logical space into possibilities and choose between them. An advocate of Open-Endedness is someone who thinks that there is no such thing as an exhaustive collection of distinctions: a collection of distinctions to which every distinction reducible. For whenever you give me some distinctions, I can use them to characterize further distinctions.[10] In slogan form: there is no such thing as an exhaustive characterization of logical space.

Michael Dummett famously argued that certain concepts are *indefinitely extensible*.[11] My preferred way of cashing out Dummett's idea is borrowed from Uzquiano (2015*b*):

> A concept is indefinitely extensible if whenever xx are amongst its instances, xx can be used to characterize further instances.

I take the concept of possibility to be indefinitely extensible in this sense. But please keep in mind that the claim is intended not as the ascription of a shadowy nature to a range of entities ("the possibilities") but as an articulation of the thought that there is no such thing as exhaustive characterization of logical space.

I am also an *anti-absolutist* about possibility: I think there is no sense to be made of "absolutely general" quantification over possibilities.[12] Lewis (1991) famously wondered whether an anti-absolutist about individuals would need to resort to the idea that "some mystical censor stops us from quantifying over

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[10]Here "further" means "non-reducible to the given distinctions". For distinction δ to be *reducible* to distinctions $\delta\delta$ is for it to be the case that however one wishes to take sides with respect to δ one can do so by taking sides with respect to some amongst $\delta\delta$.

[11]Dummett's discussion includes Dummett 1963, p. 195–6 and Dummett 1991, p. 316. More recent discussion includes Williamson 1998, Shapiro 2003, Shapiro & Wright 2006, and Uzquiano (2015*b*).

[12]One can treat a concept as indefinitely extensible without being an anti-absolutist with respect to that concept. (For relevant discussion, see Dummett 1991, p. 319, Dummett 1993, Hellman 2011, Linnebo 2010, Linnebo 2013, and Studd 2013.) Someone who treats the concept of possibility as indefinitely extensible but thinks there is good sense to be made of "absolutely general" quantification over possibilities is in a position to give a model of the open-endedness of logical space that is very different from the one provided here. Humberstone (1981), for example, suggests a semantics for modal languages that is based on a collection of non-maximally-specific possibilities and uses non-standard clauses for the logical connectives and modal operators. There is a lot of interesting work that can be done using approaches of this kind. (See, for instance, Rumfitt 2012, Holliday 2015, Linnebo typescript.) My understanding of these issues was greatly improved by an anonymous referee, to whom I am most grateful.

absolutely everything without restriction" (p. 68). Transposed to the present context, Lewis's quip would be profoundly misleading. For it would suggest that our anti-absolutist is prepared to grant that there is sense to be made of an "absolutely general" domain of possibilities, while objecting only to the further claim that our quantifiers can be stretched wide enough to encompass all of the domain at once. But, according to the anti-absolutist, this is exactly the wrong picture. Quantifying over possibilities is not a matter of starting with the "absolutely general" domain and shaving off a limited subdomain—perhaps a subdomain tailored to satisfy Lewis's mystical censor. The anti-absolutist thinks that we have no choice but to build our domain of quantification from the ground up, by characterizing the possibilities we intend to talk about. And, as we have seen, characterizing some possibilities creates the materials to build further possibilities. So there is a process of domain expansion that cannot be exhausted. It is for this reason, and not because of a mystical censor, that there is no sense to be made of an "absolutely general" quantification over possibilities.[13] It is also for this reason that I take the concept of possibility to be indefinitely extensible.

5 How to Be a Modal Logicist

We have considered two conceptions of metaphysical possibility: Lewisianism and necessitism. Either of them could be construed as an instance of modal logicism by adjusting the notion of absurdity to fit its contours. But neither is a natural candidate for the job. For each of them is committed to imposing *fixed bounds* on the metaphysical possibilities. Lewisianism imposes such bounds by placing limits on the size of possible spacetime; necessitism does so by demanding that every metaphysical possibility concern the individuals in a fixed totality. But we saw in §3 that logical space is open-ended, given natural assumptions about the limits of absurdity. So any conception of metaphysical possibility that imposes fixed bounds on the metaphysical possibilities will fall short of modal logicism on pain of violating those assumptions.

If one wants to develop a logicist conception of metaphysical possibility, a more natural candidate for the job is the view I shall refer to as *Logical Contingentistm*.

[13]For a more detailed characterization of anti-absolutism as it applies to individuals, see Rayo 2017.

Logical Contingentism is a version of modal logicism that draws its inspiration from the conception of metaphysical possibility of Stalnaker (2012). It begins with the idea that there is no such thing as a *canonical* partition of logical space—a partition that deserves a place in our theoretical lives independently of our particular circumstances. We have no choice but to partition logical space using the objects and properties we happen to have access to, in accordance with our particular theoretical aims. As our conception of the world evolves, we may change our views about which objects exist and which properties make sense, and thereby change the way in which we partition logical space. And as our theoretical aims evolve, we may feel the need to refine our partition by introducing possibilities that take a stand with respect to additional questions.

To this initial picture, the logical contingentist adds a rejection of necessitism. On the one hand, she claims that there are contingently existing individuals (e.g. I might have failed to exist); on the other, she claims that there might have been individuals that do not, in fact, exist (e.g. had I had a sister, she wouldn't have been identical to anything there, in fact, is).

The rejection of necessitism has two important consequences for the logical contingentist. First, it helps her hold onto her modal logicism without resorting a revisionist conception of absurdity. For there is no obvious absurdity in a way for the world to be on which I fail to exist, or one on which I have a sister who does not, in fact, exist. Second, and more importantly, it means that the logical contingentist takes the question of what possibilities make sense to have an answer that is only contingently true. Since I exist, I am available to characterize various possibilities: for instance, the possibility that I am now wearing a hat. But I do not have a sister. So "she" is not available to characterize a way for the world to be on which "she" is a plumber. There is therefore no sense to be made of such a possibility. But had I had a sister, she would have been available to characterize a way for the world to be on which *she herself* is a plumber and there would have been good sense to be made of such a possibility.

The logical contingentist takes metaphysical possibility to be open-ended in two different ways. First, she is a modal logicist. So, assuming she embraces a sufficiently natural conception of absurdity, she will take the argument in §3 to show that there is no such thing as a space of maximally specific metaphysical possibilities. Second, the logical contingentist thinks that metaphysical possibility is *potentially* open-ended: she thinks that there might have been sense to be made of possibilities that do not, in fact, make sense. For she thinks that there might have been individuals that do not, in fact, exist. And she thinks that if there had been such individuals—if I had had a sister, for example—they could have been used to make sense of possibilities that do not, in fact, make sense.

6 A Problem for Logical Contingentists

I take logical contingentism to be an attractive conception of metaphysical possibility, and therefore an attractive way of spelling out modal logicism. But it faces an important challenge. In this section I will explain what the challenge is; in the next I will suggest a way forward.

On the standard way of developing a possible-worlds semantics for modal languages (Kripke 1963), "$\Diamond$" is taken to range over the possibilities in a partition of logical space. These possibilities are usually described as "worlds", but they need not be maximally specific: they need only be specific enough to settle any questions expressible in the language.

The problem I would like to discuss is due to McMichael (1983) and arises from the fact that the logical contingentist is not in a position to specify an acceptable range for "$\Diamond$". Consider, for example, the following modal sentence:

(S) $\Diamond\exists x(\text{Sister}(x) \wedge \text{Poet}(x) \wedge \Diamond\text{Plumber}(x))$

There might have been someone who was my sister and a poet but who might have instead been a plumber. (I shall assume that plumbers can't be poets, to simplify the exposition.)

It is natural to think that (S) is true. But in order for (S) to be counted as true on a standard semantics, the range of "$\Diamond$" must include possibilities π_0 and π_1 such that:

1. according to π_0, someone is my sister and a poet
2. according to π_1, someone is a plumber rather than a poet
3. conditions 1 and 2 are witnessed by the same individual

From the point of view of a necessitist, it is clear that there are possibilities meeting all three of these conditions. For the necessitist thinks that there is an individual s who is a "possible sister" of mine. So she thinks there is a metaphysical possibility π_0^s according to which s is my sister and a poet, and a metaphysical possibility π_1^s according to which s is a plumber. And π_0^s and π_1^s satisfy all three of the conditions above.

But it is not clear that a contingentist has access to possibilities satisfying all three conditions. For the contingentist can be expected to think that there is no such thing as a "merely possible" sister of mine: an individual who is not my sister but could have been my sister.[14] So she is not in a position to follow the necessitist in using such an individual s to characterize π_0^s and π_1^s.

One might wonder whether there is a different strategy available to the contingentist. For note that π_0^s and π_1^s are more specific than is required to satisfy condition 3, which demands only that the relevant possibilities concern the same individual—it does not impose the stricter requirement that they concern s in particular. This raises the question of whether one might instead work with a *minimal specification* of π_0 and π_1, on which they are specified richly enough to ensure that they concern the same individual but not richly enough to specify who that individual is.

Whatever its merits in other contexts, this strategy is not available to logical contingentists, or to any other contingentist who thinks that possibilities are ways for the world to be. For π_0 and π_1, minimally specified, cannot be ways for the world to be. To see this, consider the question of how the world is supposed to be according to π_0, minimally specified. Part of the answer is straightforward: someone is my sister and a poet. But π_0 is supposed to impose a further constraint. There would be no problem if the further constraint was that my sister be such that she could have been a plumber rather than a poet. For then there would be a clear answer to the question of how the world is according to π_0: I have a sister who is a poet and could have been a plumber instead. But the additional constraint is supposed to be stronger than that. It requires is that there be a sister of mine who is a poet and who, *according to* π_1, is a plumber. But all we know about π_1 is that, according to it, someone is a plumber rather than a poet and is my sister and a poet according to π_0. So it is hard to get much traction on the question what it would take for the additional constraint to be satisfied. Suppose, for example, that my parents were to have a daughter who becomes a poet. What else would be required for the additional constraint to be satisfied? That she be such that she might have been a plumber instead of a poet is not enough. That she be identical to some given individual,

[14]On the assumption that there is nothing absurd about the existence of a merely possible sister, logical contingentism is compatible with the existence of merely possible sisters. As a result, the logical contingentist might accept the claim that I have no merely possible sisters but treat is as contingent. Alternatively, she might choose to rule out merely possible sisters as absurd and thereby deligitimize a family of awkward theoretical questions (e.g. how many merely possible sisters do I have, and why don't I have more?).

s, who might have been a plumber instead of a poet is too much. It is not clear that there is much room in between.

7 Clausal Semantics

The moral of the previous section is that the logical contingentist is not in a position to adopt a standard possible-worlds semantics for modal sentences. For she is committed to thinking that there is no sense to be made of the possibilities that would be needed to count a sentence like (S) as true. Notice, however, that the logical contingentist also thinks that there *would* have been sense to be made of suitable possibilities had matters been otherwise. Had I had a sister, there would have been sense to be made of a possibility at which *she* is my sister and a poet, and of a possibility at which *she* is a plumber rather than a poet.

This suggests a way forward for the logical contingentist. She might propose a semantics on which certain occurrences of "$\Diamond$" range over possibilities that make sense within a suitable modal clause (e.g. "I might have had a sister such that ..."), even if they wouldn't make sense outside the relevant clause. For example, she might take

(S) $\Diamond\exists x(\text{Sister}(x) \wedge \text{Poet}(x) \wedge \Diamond\text{Plumber}(x))$

to have the following truth-conditions, where $\pi\pi$ constitute a partition of logical space:

(S_{tc}) At some possibility among $\pi\pi$, there is an individual x who is my sister and a poet, and who is such that, at some possibility among $\pi\pi_x$, she is a plumber rather than a poet.

Here the possibilities $\pi\pi_x$ are the result of refining $\pi\pi$ by having them coherently take sides on the questions whether x exists, whether she is my sister, and whether she is a poet or a plumber. Note that the possibilities $\pi\pi_x$ only make sense with respect to a given individual x. And since I do not, in fact, have a sister, there is no way of specifying an appropriate value for "x" independently of a modal clause. But "$\pi\pi_x$" is legitimately used in (S_{sc}) because it occurs within the scope of a suitable modal clause ("at some possibility among $\pi\pi$, there is an individual x").

In Appendix B, I generalize this approach to develop a "clausal semantics" for a first-order modal language $L^{\Diamond}$ (with or without plural variables and quantifiers). The basic idea is straightforward. One starts by indexing each occurrence of "$\Diamond$" with the list of variables that occur free in its scope.[15] Each unindexed occurrence of "$\Diamond$" is taken to range over a partition of logical space, $\pi\pi$. But each indexed occurrence, $\ulcorner\Diamond_{x_1,\ldots,x_k}\urcorner$, is evaluated within the scope of a suitable modal clause, and is taken to range over the possibilities $\pi\pi_{x_1,\ldots,x_k}$, which result from refining $\pi\pi$ by having them coherently take sides on the existence and properties of $x_1, \ldots, x_k$.

Consider, for instance,

(C) $\Diamond\exists x\Diamond\exists y\neg\Diamond\exists z\exists w(x = z \wedge y = w)$

There might have been an x such that there might have been a y such that it is impossible for both x and y to exist.

It is indexed as follows:

$$\Diamond\exists x\Diamond_x\exists y\neg\Diamond_{x,y}\exists z\exists w(x = z \wedge y = w)$$

and assigned the following truth-conditions:

(C_{tc}) At some possibility among $\pi\pi$, something x is such that at some possibility among $\pi\pi_x$, something y is such that at no possibility among $\pi\pi_{x,y}$, both x and y exist.

This example illustrates an important feature of the proposal. As usual, there is only sense to be made of the possibilities $\pi\pi_{x,y}$ with respect to given x and y. But note that if (C) is true, there is no *possibility* with respect to which suitable x and y exist, and therefore no *possibility* with respect to which there is sense to be made of $\pi\pi_{x,y}$ (independently of modal clauses). So: from which point of view are we to make sense of $\pi\pi_{x,y}$? Answer: from a point of view that we, the theorists, are able to generate from our present context, using suitable modal clauses.

Stalnaker (2012, Appendix B) develops a semantics for modal sentences that uses higher-level propositional functions instead of the modal clauses I am recommending here. As Jacinto (2019) points out, however, it is hard to spell out the resulting semantics without using ingredients that a contingentist is not in a position to take at face value. Stalnaker concludes from Jacinto's observation that "virtual propositional functions [which are artifacts of the model that do not correspond to anything in reality] seem to play an ineliminable role in the compositional process by which complex predicates and quantificational

[15] I assume that "$\Box$" is defined in terms of "$\Diamond$", in the usual way.

constructions are interpreted" (Stalnaker 2016). A nice feature of the clausal semantics is that it relies exclusively on entities that the logical contingentist is in a position to take at face value.[16]

8 Model Theory

In Appendix C, I develop a model-theory for the clausal semantics of Appendix B and use it to show that there is a precise sense in which a clausal semantics delivers the same results as a standard (Kripkean) possible-worlds semantics.

The easiest way to explain how the model theory works is to see it from the perspective of a necessitist, who thinks there are some objects, tt, such that, necessarily, everything among them. Our necessitist also thinks that tt can be used to characterize a "canonical" partition of logical space, $\mathcal{W}^{tt}$, and that arbitrary occurrences of "$\Diamond$" should be taken to range over the worlds in $\mathcal{W}^{tt}$, in accordance with a standard possible-worlds semantics.

Notice, however, that our necessitist can use $\mathcal{W}^{tt}$ to model more limited perspectives than her own. In particular for each subplurality xx of tt, she can use $\mathcal{W}^{tt}$ to model the perspective of someone who is only in a position to make distinctions based on the vocabulary of the object-language, $L^{\Diamond}$, and the individuals in xx. More specifically, she characterizes an equivalence relation $\sim_{xx}$ amongst worlds in $\mathcal{W}^{tt}$ that satisfies the following condition: $w \sim_{xx} v$ iff w and v cannot be discriminated using only the vocabulary of $L^{\Diamond}$ and individuals in xx.[17] She then defines a *level-xx proposition* as a subcollection of $\mathcal{W}^{tt}$ that includes both or neither of every pair of $\sim_{xx}$-equivalent worlds. Accordingly, she can think of a level-xx proposition as a proposition that is available from the perspective of someone who speaks $L^{\Diamond}$ and is able to reference xx.

The necessitist sees the contingentist as failing to recognize the existence of certain individuals among tt: my possible sisters, for example. She can model this limited perspective as the claim that there is only sense to be made of a level-xx proposition when xx are among the individuals that contingentists take

[16] As Peter Fritz pointed out to me, it is not difficult to characterize a homophonic semantics for first-order modal languages in a higher-order λ-calculus, provided that one is not committed to the Being Constraint of Williamson 2013. An advantage of the semantics proposed here is that it is compatible with Williamson's Being Constraint, which is embodies in the semantic clauses of Appendix B.

[17] The basic idea is due to Stalnaker 2012, Appendix A and is further developed in Fritz 2016.

to exist. More generally, the necessitist can model the contingentist's indexed modal operator "$\Diamond_{v_1,\ldots,v_k}$" (which is meant to be evaluated within a suitable modal clause) as ranging over xx-level propositions (where xx are the denotations assigned to $v_1, \ldots, v_k$ within the relevant clause). This allows her to prove the following results:

> *Notation:*
>
> I assume, for simplicity, that $\mathcal{W}^{tt}$ is a set, and that a proposition is a subset of $\mathcal{W}^{tt}$. The propositions $[\![\phi]\!]^\sigma$ and $[\![\phi]\!]^\sigma_K$ are then defined as follows:
>
> $[\![\phi]\!]^\sigma =$ the set of worlds $w \in \mathcal{W}^{tt}$ such that ϕ is true at w (relative to variable assignment σ) according to the "clausal" model theory of Appendix C.
>
> $[\![\phi]\!]^\sigma_K =$ the set of worlds $w \in \mathcal{W}^{tt}$ such that ϕ is true at w (relative to variable assignment σ) according to a standard (Kripkean) model theory.

Level Theorem $[\![\phi]\!]^\sigma$ is a level-$\emptyset$ proposition (ϕ a setnence, σ an assignment).[18]

Equivalence Theorem $[\![\phi]\!]^\sigma = [\![\phi]\!]^\sigma_K$ (ϕ a formula, σ an assignment).

Validity Theorem A sentence ϕ is valid with respect to a "clausal" semantics just in case it is valid with respect to a standard Kripke-semantics (assuming a trivial accessibility relation).

The upshot of the Level Theorem is that every proposition expressed by a sentence of $L^\Diamond$ is a proposition that the logical contingentist takes to make sense, assuming she is able to make sense of the vocabulary in $L^\Diamond$. So even though the necessitist thinks that the totality of possible individuals, *tt*, can be used characterize subsets of $\mathcal{W}^{tt}$ that the contingentist would regard as nonsense (such as the proposition, concerning a possible sister of mine, that she be a plumber), such propositions can never be asserted by speakers of $L^\Diamond$. So the logical contingentist can regard $L^\Diamond$ as a safeguard against nonsense: any claim that the necessitist is able to state in $L^\Diamond$ is a claim that the contingentist is able to make sense of.

[18] A level-$\emptyset$ proposition is a proposition that never separates $\sim_\emptyset$-equivalent worlds: worlds that cannot be discriminated using only distinctions based on the predicates of $L^\Diamond$.

The upshot of the Equivalence Theorem is that a clausal semantics and a standard possible-worlds semantics yield identical results. A clausal semantics must follow a circuitous path in order to avoid making use of resources that the logical contingentist treats as illegitimate. But it ultimately delivers the same assignment of propositions to sentences (and the same assignment of propositions to formulas, relative to variable assignments). So logical contingentists can feel free to use a standard semantics when convenient, without having to worry too much about its ontological profligacy. As long as they keep in mind that the official story must ultimately be based on the more cumbersome (but ontologically restrained) clausal semantics, they can feel confident that the standard semantics won't leave them astray.

The Validity Theorem is an easy corollary of the Equivalence Theorem and reassures the logical contingentist that she need not revise her modal logic to accommodate her philosophy of modality.

9 Fritz's Puzzle

Fritz's Puzzle (Fritz 2017) is the observation that each of the following claims is initially plausible even though it is equivalent to the other's negation.[19]

(F1) $\Box\forall xx\,(I(xx) \rightarrow \Diamond\exists y(I(y) \wedge y \not\prec xx))$

Whichever island universes there might have been, there might have been another.

(F2) $\Diamond\exists xx\,(I(xx) \wedge \Box\forall y(I(y) \rightarrow y \prec xx))$

There might have been some island universes that included all possible island universes.

Since (F1) and (F2) are equivalent to each other's negations, exactly one of them is true. But which one? From the point of view of a necessitist, this is an awkward question. For the necessitist thinks there is a definite totality,

[19]"$I(\ldots)$" applies to island universes, in the sense of §3, and "$I(xx)$" is a syntactic abbreviation for "$\forall z(z \prec xx \rightarrow I(z))$". Fritz restricts the quantifiers to angels rather than island universes, following Hawthorne & Uzquiano 2011. All that matters for present purposes is that one use a category of individuals such that it is natural to suppose that the existence of some individuals in that category imposes no restrictions on the existence of other individuals in that category.

ii, of possible island universes.[20] So she sees the choice between (F1) and (F2) as tantamount to the question whether ii might have all been island universes together. If the answer is "no", one can follow up by asking, "why not?". If the answer is "yes", one can follow up by asking of some objects xx equinumerous with ii, "why couldn't have there been more island universes than there are xx?". It is hard to see how the necessitist could give satisfying answers to either of these followup questions.

The logical contingentist is also forced to choose between (F1) and (F2). But, unlike the necessitist, she can make her choice in a way that minimizes awkward followup questions. For suppose the contingentist accepts (F1). In other words: suppose she thinks that whichever island universes there might have been, there might have been another. Our contingentist faces no pressure to answer the follow-up question "why couldn't all possible island universes existed together?" because, unlike the necessitist, she is in a position to deny that there is sense to be made of "all possible island universes".

In fact, the logical contingentist can use the open-endedness argument of §3 to make a positive case for (F1):

> Suppose, for *reductio*, that (F2) is true. Then there must be a possibility, π, according to which there are some island universes, ii, which include all possible island universes. But—working from the perspective of π—one can use the open-endedness argument of §3 to specify a way for the world to be on which there are more island universes than there are ii. Since such a way for the world to be involves no absurdity, it contradicts the assumption that ii include all possible island universes.

(I give a formal counterpart of this argument in Appendix C.6 by noting that the logical contingentist should restrict attention to models that are "indefinitely extensible" with respect to *I* and observing that such models always verify (F1).)

Fritz's Puzzle illustrates the fact that the necessitist and the logical contingentist face different explanatory demands. The necessitist is committed to thinking that there is sense to be made of "all possible island universes", and therefore faces pressure to explain whether such individuals could have all been

[20]She thinks that, necessarily, every island universe, in fact, exists, and therefore that ii are just those amongst the individuals that, in fact, exist that enjoy the property of being such that they could have been an island universe.

island universes together, and why. This is an awkward explanatory burden because it is hard to see how it could give rise to a fruitful theoretical enterprise. The logical contingentist, in contrast, is in a position to avoid such awkwardness by denying that there is good sense to be made of "all possible island universes".

10 Concluding Remarks

Bob Stalnaker once pointed out to me that a problem with necessitism is that it is committed to thinking that there are definite answers to questions that shouldn't have answers. I take one such question to be the question of how many possible bald men in the doorway there are (Quine 1948), and another to be the question of how many possible island universes there are (and whether they could all be island universes together).[21] From the point of view of a logical contingentist, in contrast, the offending questions can be rejected outright. Since there is no sense to be made of "all possible island universes", there is no need to answer the question of how many of "them" there are (or the question of whether "they" could have all been island universes together).

This suggests a general strategy for adjudicating the debate between necessitists and logical contingentists. Rather than construing the debate as a disagreement about ontology (e.g. a disagreement about the individuals that ought to be included in the range of an "absolutely general" quantifier), it can be construed as an effort to constrain the range of questions we treat as legitimate. On the one hand, we want to count enough questions as legitimate to allow for a robust body of theory; on the other, we want to rule out enough questions as illegitimate to minimize specious research programs.

We have seen that the range of questions that necessitists treats as legitimate has advantages and disadvantages. On the one hand, the necessitist thinks there is sense to be made of a totality, ss, of "all possible sisters of mine". And, as we saw in §6, this has the advantage of allowing us consider the question, concerning an individual among ss, whether *she* might have been a poet or a plumber, and use the resulting answer to characterize the sorts of possibilities that are needed to construct a standard possible-worlds semantics for modal languages. On the other hand, the necessitist is saddled with awkward questions: How many possible island universes are there? Could they all have all been island universes together?

[21]For related discussion, see Hawthorne & Uzquiano 2011.

The range of questions that logical contingentists treat as legitimate has a different set of advantages and disadvantages. It has the advantage of not having to answer the question of how many possible island universes there are, or whether they could have all been island universes together. But it has the disadvantage of making it hard for the logical contingentist to take a standard possible-worlds semantics for modal languages at face value.

This paper can be seen as making a (partial) case for the view that logical contingentism delivers a better package, all things considered. For even though the disadvantages are real, some of them can be ameliorated. As we saw in §§7–8, the logical contingentist is in a position to specify truth-conditions for modal sentences using a "clausal" semantics, which can be shown to deliver the same results as a standard possible-worlds semantics.

I certainly don't mean to suggest that this settles the debate between contingentists and necessitists. Firstly, I have restricted attention to a fairly narrow set of issues. For example, I have not considered the question of whether the clausal semantics of §7 could be extended to a natural language with different kinds of modalities (Kratzer 1991). Secondly, I doubt there is a neutral way of drawing the line between fruitful and spurious theoretical projects. So we shouldn't expect the debate between logical contingentists and necessitists to be decided by a knock-down argument.[22]

[22]For their many helpful comments, I am grateful to Arif Ahmed, Nathaniel Baron–Schmitt, Matti Eklund, Adam Elga, Jeremy Goodman, Caspar Hare, Nick Jones, Øystein Linnebo, Sam Roberts, Jonathan Shaheen, Jack Spencer, Gabriel Uzquiano, Robbie Williams, and two anonymous referees. I am also grateful to audiences at Dartmouth University, Simon Fraser University, the University of Miami, the University of Oslo, the University of St Andrews, the University of Uppsala, and the 2017 meeting of the American Association of Mexican Philosophers at UC, Boulder. Special thanks are due to Bruno Jacinto, Cian Dorr, Peter Fritz, and Bob Stalnaker.

Appendices

A Higher-Order Languages

In the main text, I simplify the exposition by using colloquial language to talk about possibilities. The proper way of proceeding is to use a higher-order language instead: specifically, a simple typed language with lambda abstraction. Such a language variables of type $\langle\rangle$ (i.e. propositional variables) and a relation "$\equiv$" of type $\langle\langle\rangle,\langle\rangle\rangle$ that expresses the analogue of identity for type $\langle\rangle$. I assume that "$\equiv$" is an equivalence relation and that its interaction with the logical connectives is Boolean, in the sense of Dorr 2017. "$\gg$" expresses entailment for type $\langle\rangle$ and can be defined as follows:

$$p \gg q =_{df} p \equiv (p \wedge q)$$

Intuitively, we think of the variables of type $\langle\rangle$ as expressing ways for the world to be, thereby equating propositions with ways for the world to be. $\top^{\langle\rangle}$ is a trivial way for the world to be; it is assumed to satisfy the triviality principle $\forall q^{\langle\rangle}(q \gg \top)$ and (therefore) the uniqueness principle $\forall q^{\langle\rangle}(\top \gg q \rightarrow q \equiv \top)$. Similarly, $\bot^{\langle\rangle}$ is an absurd way for the world to be; it is assumed to satisfy the explosion principle $\forall q^{\langle\rangle}(\bot \gg q)$ and (therefore) the uniqueness principle $\forall q^{\langle\rangle}(q \gg \bot \rightarrow q \equiv \bot)$. A possibility is a way for the world to be other than $\bot$. Formally:

$$\text{Possibility} =_{df} \lambda p^{\langle\rangle}.p \not\equiv \bot$$

$$\Diamond(\phi) =_{df} \text{Possibility}(\phi)$$

In addition, we assume that our typed language has been enriched with "plural" types: ee, $\langle\rangle\langle\rangle$, etc, and with a family of relations $\prec$, one for each singular type τ, which hold of a variable of type τ and a variable of corresponding plural type, $\tau\tau$. Intuitively, $\prec$ expresses the plural analogue of membership. So, for instance, the colloquial expression "way for the world to be ... is among ways for the world to be ..." might be formalized as:

$$\lambda p^{\langle\rangle} pp^{\langle\langle\rangle\rangle} \, . \; p \prec pp$$

Similarly the colloquial predicate "... constitute a partition of logical space" can be characterized as follows:

$$\text{Partition} =_{df} \lambda pp^{\langle\rangle\langle\rangle} \, . \, \Diamond(pp) \wedge \text{ Exhaustive}(pp) \wedge \text{Non-Overlapping}(pp)$$

where:

$$\Diamond(pp) =_{df} \forall p^{\langle\rangle}(p \prec pp \to \Diamond p)$$

$$\text{Exhaustive} =_{df} \lambda pp^{\langle\rangle\langle\rangle} \,.\, \forall p^{\langle\rangle}(\forall q^{\langle\rangle}(q \prec pp \to q \gg p) \to p \equiv \top)$$

$$\text{Non-Overlapping} =_{df} \lambda pp^{\langle\rangle\langle\rangle}.\forall p^{\langle\rangle}\forall q^{\langle\rangle}((p \prec pp \wedge q \prec pp) \to (p \wedge q) \equiv \bot)$$

The colloquial claim labelled Open-Endedness of §3—there is no such thing as a maximally specific partition of logical space—can be cashed out as:

$$\text{Open-Endedness} =_{df} \forall pp^{\langle\rangle\langle\rangle}(\text{Partition}(pp) \to \exists p^{\langle\rangle}(p \prec pp \wedge \neg \text{FullyDetermines}(p)))$$

where "FullyDetermines" is a predicate of type $\langle\rangle$ that is true of propositions that fully determine how the world is. If one were entitled to assume that type $\langle\rangle$ quantification is "absolutely general", then "FullyDetermines" could be explicitly defined as:

$$\lambda p^{\langle\rangle}.\neg(p \equiv \bot) \wedge \forall q^{\langle\rangle}(p \gg q \vee p \gg \neg q)$$

In the present context, however, we are not entitled to that assumption. So "FullyDetermines" must be regarded as primitive.

The argument for Open-Endedness of footnote 8 assumes the following well-ordering principle:

Well-Ordering Principle $\exists <^{\langle e,e\rangle}$ Well-Order$(<, xx)$

where "WellOrder" is defined in the obvious way:

$$\begin{aligned} \text{WellOrder} \;\; =_{df} \;\; \lambda R^{\langle ee\rangle}\lambda xx^{ee} \,.\, & \\ & \forall x, y \prec xx(Rx, y \to \neg Ry, x)\wedge \\ & \forall x, y, z \prec xx(Rx, y \wedge Ry, z \to Rx, z)\wedge \\ & \forall x, y \prec xx(x \neq y \to Rx, y \vee Ry, x)\wedge \\ & \forall yy \preceq xx(\exists y(y \prec yy) \to \exists y \prec yy \forall x \prec yy(x \neq y \to Rx, y)) \end{aligned}$$

The argument of footnote 8 also presupposes a few additional definitions. Intuitively, "$yy \approx zz$" says that yy are just as many as zz, and "$yy \approx^* zz$" says that there are just as many yy as there are subpluralities of zz. Formally:

$$\begin{aligned} \approx \;\; =_{df} \;\; \lambda yy^{ee}\lambda zz^{ee}.\exists R^{\langle e,e\rangle}(\;\; & \forall y \prec yy\; \exists! z \prec zz(R(y, zz))\wedge \\ & \forall z \prec zz\; \exists! y \prec yy(R(y, zz)) \;\;) \end{aligned}$$

$$\begin{aligned} \approx^* \;\; =_{df} \;\; \lambda yy^{ee}\lambda zz^{ee}.\exists R^{\langle e,ee\rangle}(\;\; & \forall y \prec yy\; \exists! ww \preceq zz(R(y, ww))\wedge \\ & \forall ww \preceq zz\; \exists! y \prec yy(R(y, ww)) \;\;) \end{aligned}$$

With these definitions in place, it is easy to characterize f, for fixed xx. By the Well-Ordering Principle, there is a relation $<$ such that Well-Order($<, xx$). That relation can be used to give the following recursive definition:

- $f(0) =_{df} \forall zz(\forall z(Az \leftrightarrow z \prec zz) \rightarrow zz \approx \mathbb{N})$

  where 0 is the $<$-least element among xx.

- $f(x') =_{df} \forall zz\,[\forall z(Az \leftrightarrow z \prec zz) \rightarrow f(x) \gg \forall yy\,(\forall y(Ay \leftrightarrow y \prec yy) \rightarrow zz \approx^* yy)]$

  where x' is the $<$-successor of x.

- $f(\lambda) =_{df} \forall zz[\quad \forall x < \lambda\, \exists ww \preceq zz(f(x) \gg \forall yy\,(\forall y(Ay \leftrightarrow y \prec yy) \rightarrow ww \approx ww)) \rightarrow$
  $\forall yy(\forall y(Ay \leftrightarrow y \prec yy) \rightarrow zz \approx yy) \quad]$

  where λ is distinct from 0 and has no $<$-predecessor among xx.

To verify that our recursive clauses characterize a unique function on xx, suppose for *reductio* that there is a function f' that satisfies the recursive clauses and is such that $f(x) \neq f'(x)$ for some $x \prec xx$. Let $z \prec xx$ be the $<$-smallest such x. If $z = 0$, the result is immediate. If z has an immediate $>$-predecessor, y, the result follows from the observation that $f(y) = f'(y)$. If z has no immediate $>$-predecessor, the result follows from the observation that $f(y) = f'(y)$ for every $y < z$.

The plural generalization of Cantor's Theorem that footnote 8 alludes to is:

Cantor's Theorem (plural) $\exists x, y(x \neq y \wedge x \prec xx \wedge y \prec xx) \rightarrow xx \not\approx^* xx$.

versions of which are proved in Bernays 1942, 137–8 and Shapiro 1991, Theorem 5.3. This result immediately entails that there are more island universes according to $f(x')$ than according to $f(x)$ (and therefore that there are more island universes according to $f(\lambda)$ than according to any of its predecessors, for λ a $<$-limit). So $f(x)$ and $f(y)$ must be incompatible whenever $x \neq y$.

Like colloquial talk of ways for the world to be, colloquial talk of distinctions between ways for the world to be should be understood in higher-order terms. The colloquial predicate "…is a distinction between coherent ways for the world to be" should be replaced by the higher-order predicate "D", which is defined as follows:

$$\lambda\delta^{\langle\langle\rangle\rangle}.\exists p^{\langle\rangle}(\delta(p) \wedge \forall q^{\langle\rangle}(\delta(q) \leftrightarrow (q \equiv p \vee q \equiv \neg p)) \wedge \neg(p \equiv \top \vee p \equiv \bot))$$

B A Compositional Semantics for the Logical Contingentist

Let $L^{\Diamond}$ be a first-order language with identity, which has been enriched with the modal operator "$\Diamond$", plural variables and quantifiers, the plural inclusion predicate "$\prec$", and the plural identity predicate "$=$". To keep things simple, I shall assume that $L^{\Diamond}$ contains no function-letters or individual constants. I shall also assume that each n-place predicate $\ulcorner \mathrm{P}_i^n \urcorner$ of $L^{\Diamond}$ can be translated into the metalanguage using the (Italicized) metalinguistic predicate $\ulcorner P_i^n \urcorner$.

Let the "base" possibilities $\pi\pi$ constitute a partition of logical space that the logical contingent is in a position to make sense of. (To keep things simple, I shall assume that they form a set.) A proposition is a set of possibilities from among $\pi\pi$. A semantics for $L^{\Diamond}$ is a compositionally specified function $[\![\ldots]\!]$ that assigns a proposition to each sentence of $L^{\Diamond}$.

We begin with some preliminary notation:

- $\ulcorner \vec{x_k} \urcorner$ abbreviates $\ulcorner x_1, \ldots, x_k \urcorner$.
- $\ulcorner x\vec{x}_l \urcorner$ abbreviates $\ulcorner xx_1, \ldots, xx_l \urcorner$.
- For any given $x_1, \ldots, x_k, xx_1, \ldots, xx_l$, the possibilities $\pi\pi_{\vec{x_k}, x\vec{x}_l}$ are the result of refining the base possibilities $\pi\pi$ by having them coherently take sides on the existence and properties of each individual in $x_1, \ldots, x_k$ and each individual among $xx_1, \ldots, xx_l$.
- For a given individual x, we say that possibility π is x-refined by possibility π' (in symbols: $\pi \rhd_x \pi'$) iff π' is a refinement of π that coherently settles the question of x's existence in the affirmative.
- For some given individuals xx, we say that possibility π is xx-refined by possibility π' (in symbols: $\pi \rhd_{xx} \pi'$) iff π' is a refinement of π that coherently settles the question of the existence of each xx in the affirmative.
- Since $\pi\pi$ constitute a partition of logical space, exactly one of them is actualized. We shall call this possibility "$\pi_@$" and let its domain be $D_@$:

$$D_@ = \{z : \text{at } \pi_@, \exists x(x = z)\}$$

- $D_{\vec{x_k}, x\vec{x}_l} = D_@ \cup \{x_1, \ldots, x_k\} \cup \{z : z \prec xx_1\} \cup \cdots \cup \{z : z \prec xx_l\}$.

- We let $\Sigma_{\vec{x_k},x\vec{x}_l}$ be the set of (partial) variable assignments into $D_{\vec{x_k},x\vec{x}_l}$. More specifically, where the singular variables of $L^{\Diamond}$ are $u_1, u_2, \dots$ and its plural variables are $uu_1, uu_2, \dots,$

$$\Sigma_{\vec{x_k},x\vec{x}_l} = \left\{\sigma : \begin{array}{l}\sigma \text{ is a partial function from } \{u_1, u_2, \dots\} \text{ into } D_{\vec{x_k}} \\ \text{and from } \{uu_1, uu_2, \dots\} \text{ into } \wp(D_{\vec{x_k}}) - \emptyset.\end{array}\right\}$$

Σ is the special case of $\Sigma_{\vec{x_k}}$ in which the sequence $\vec{x_k}$ is empty:

$$\Sigma = \left\{\sigma : \begin{array}{l}\sigma \text{ is a partial function from } \{u_1, u_2, \dots\} \text{ into } D_{@} \\ \text{and from } \{uu_1, uu_2, \dots\} \text{ into } \wp(D_{@}) - \emptyset.\end{array}\right\}$$

Note that when $D_{@}$ is empty, Σ consists of the partial function on $\{u_1, u_2, \dots\}$ that is everywhere undefined.

- For $\sigma \in \Sigma_{\vec{x_k},x\vec{x}_l}$ and $z \in D_{\vec{x_{k+1}},x\vec{x}_l}$, $\sigma[z/v] \in \Sigma_{\vec{x_{k+1}}}$ is the variable assignment that assigns z to v and is otherwise like σ.

- For $\sigma \in \Sigma_{\vec{x_k},x\vec{x}_l}$ and $\{z : z \prec zz\} \subseteq D_{\vec{x_n},x\vec{x_{l+1}}}$, $\sigma[zz/vv] \in \Sigma_{\vec{x_k},x\vec{x_{l+1}}}$ is the variable assignment that assigns zz to vv and is otherwise like σ.

Next, for $\sigma \in \Sigma_{\vec{x_k},x\vec{x}_l}$ and $\pi \prec \pi\pi_{\vec{x_k},x\vec{x}_l}$, we characterize the function $[\![\dots]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l}$, which assigns to each formula of $\mathcal{L}$ a set of possibilities among $\pi\pi_{\vec{x_k},x\vec{x}_l}$. (Note that $[\![\dots]\!]^{\sigma}$ is the special case of $[\![\dots]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l}$ in which $\vec{x_k}$ and $x\vec{x}_l$ are both empty.)

- $[\![u_i = u_j]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} = \{\pi \in \pi\pi_{\vec{x_k},x\vec{x}_l} : \text{at } \pi, \exists z(z = \sigma(u_i) \wedge z = \sigma(u_j))\}$

- $[\![uu_i = uu_j]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} = \{\pi \in \pi\pi_{\vec{x_k},x\vec{x}_l} : \text{at } \pi, \exists zz(zz = \sigma(uu_i) \wedge zz = \sigma(uu_j))\}$

- $[\![u_i \prec uu_j]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} = \{\pi \in \pi\pi_{\vec{x_k},x\vec{x}_l} : \text{at } \pi, \exists z\exists zz(z = \sigma(u_i) \wedge zz = \sigma(uu_j) \wedge z \prec zz)\}$

- $[\![\mathrm{P}^n_i(u_1, \dots, u_n)]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} =$

  $\{\pi \in \pi\pi_{\vec{x_k},x\vec{x}_l} : \text{at } \pi, \exists z_1 \dots \exists z_n(z_1 = \sigma(u_1) \wedge \dots \wedge z_n = \sigma(u_n) \wedge P^n_i(z_1, \dots, z_n))\}$

- $[\![\exists v\phi]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} =$

  $\left\{\pi \in \pi\pi_{\vec{x_k},x\vec{x}_l} : \text{ at } \pi, \exists x_{k+1} \left(\left\{\pi' \in \pi\pi_{\vec{x_{k+1}},x\vec{x}_l} : \pi \rhd_{x_{k+1}} \pi' \wedge \pi' \in [\![\phi]\!]^{\sigma[x_{k+1}/v]}_{\vec{x_{k+1}},x\vec{x}_l}\right\} \neq \emptyset\right)\right\}$

- $[\![\exists vv\phi]\!]^{\sigma}_{\vec{x_k}} =$

  $\left\{\pi \in \pi\pi_{\vec{x_k},x\vec{x}_l} : \text{ at } \pi, \exists xx_{l+1} \left(\left\{\pi' \in \pi\pi_{\vec{x_k},x\vec{x_{l+1}}} : \pi \rhd_{xx_{l+1}} \pi' \wedge \pi' \in [\![\phi]\!]^{\sigma[xx_{l+1}/vv]}_{\vec{x_k},x\vec{x_{l+1}}}\right\} \neq \emptyset\right)\right\}$

- $[\![\neg\phi]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} = (\pi\pi_{\vec{x_k},x\vec{x}_l}) - [\![\phi]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l}$
- $[\![(\phi \wedge \psi)]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} = [\![\phi]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} \cap [\![\psi]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l}$
- $[\![\Diamond\phi]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} = \begin{cases} \pi\pi_{\vec{x_k},x\vec{x}_l}, \text{ if } [\![\phi]\!]^{\sigma}_{\vec{x_k},x\vec{x}_l} \neq \emptyset \\ \bot, \text{ otherwise} \end{cases}$

Finally, for ϕ a sentence of $L^{\Diamond}$, we let

$$[\![\phi]\!] = \{\pi \in \pi\pi : \text{for any } \sigma \in \Sigma, \pi \in [\![\phi]\!]^{\sigma}\}$$

and we say that $[\![\phi]\!]$ is true iff $\pi_{@} \in [\![\phi]\!]$.

C Model Theory

I characterize a model theory for a higher-order language, L^H, which includes singular variables (of type e), plural variables (of type ee),[23] and propositional variables (of type $\langle\rangle$).

My semantics is a variation of the system proposed in Stalnaker (2012, Appendix A) and further developed in Fritz (2016). Like Stalnaker and Fritz, I work with a space of "worlds" that is divided into equivalence classes by an parameterized equivalence relation $\sim_\alpha$ meeting certain conditions. The present treatment differs from Stalnaker and Fritz's in that I allow the parameter α to vary over indices that keep track of "available distinctions" but may or may not correspond to the individuals that exist according to a particular possibility. In contrast, Stalnaker and Fritz restrict attention to the case in which α ranges over the domains of possible worlds.

[23] As I learned from Peter Fritz, an alternative is to use variables type $\langle e\rangle$ instead of plural variables, and simulate plurals using the following definitions:

$$x_i \prec xx_j =_{df} Z_j^{\langle e\rangle}(x_i)$$

$$\exists xx_i\psi =_{df} \exists Z_i^{\langle e\rangle}(\text{Rigid}(Z_i) \wedge \exists z(Z(z)) \wedge \psi$$

where

$$\text{Rigid} =_{df} \lambda Y.(\lambda z.(Yz \equiv \top \vee Yz \equiv \bot) \equiv \lambda z.z = z)$$

C.1 The object-language

Definition 1 (Definition of L^H) *L^H is a language built from the following symbols:*

- *variables of type* e*:* $x_1, x_2, \ldots,$
- *variables of type* ee*:* $xx_1, xx_2, \ldots,$
- *variables of type* $\langle\rangle$*:* $p_1, p_2, \ldots,$
- *the absurdity constant,* $\perp$*, which is of type* $\langle\rangle$*,*
- *the singular identity symbol,* $=$*, which is of type* $\langle e, e\rangle$*,*
- *for each* $k \geq 0$ *and* $n \geq 1$*, the* n*-place atomic predicate* P^n_k*, which is of type* $\Big\langle \underbrace{e, \ldots, e}_{n \text{ times}} \Big\rangle$*,*
- *the plural identity symbol,* $=$*, which is of type* $\langle ee, ee\rangle$*,*
- *the inclusion symbol,* $\prec$*, which is of type* $\langle e, ee\rangle$*,*
- *the propositional identity symbol,* $\equiv$*, which is of type* $\langle\langle\rangle, \langle\rangle\rangle$*,*
- *the existential quantifier,* $\exists$*, which binds variables of type* e *or* ee*,*
- *the relativized existential quantifier* $\underset{v_1,\ldots,v_k}{\exists}$*, which binds variables of type* $\langle\rangle$ *(where* $v_1, \ldots, v_k$ *are variables of type* e *or* ee*),*
- *the negation symbol,* $\neg$*, the conjunction symbol,* $\wedge$*, and parentheses.*

The formulas of L^H are defined recursively, in the obvious way. A sentence is a formula in which every occurrence of a variable (including quantifier subscripts) is bound by a quantifier.

Note that L^H is rich enough to interpret indexed modal formulas:

Definition 2 (Interpretation of Modal Operators) *Let ϕ be a formula of $L^\Diamond$ (which is defined in Appendix B) and let ϕ^i be the result of indexing ϕ (as explained in §7). The formula of L^H that* ***corresponds*** *to ϕ is the result of replacing each occurrence of $\Diamond_{v_1,\ldots,v_k} \ldots$ in ϕ^i by*

$$\underset{v_1,\ldots,v_k}{\exists p_i} (\neg(p_i \equiv \perp) \wedge (p_i \equiv \ldots))$$

for p_i a fresh variable.

C.2 Indices, Worlds, and Propositions

Our model theory will be based on a domain $\mathscr{D}$, whose elements are used to keep track of possible individuals. (A necessitist might take $\mathscr{D}$ to consist of the possible individuals themselves; a contingentist must use representatives to play the role of merely possible individuals.)

An **index** is a subset of $\mathscr{D}$. Indices play two distinct roles in our framework. On the one hand, an index is used to keep track of the *domain* of a given possibility (i.e. the possible individuals that exist according to that possibility). On the other hand, an index is used to keep track of a set of *distinctions* that might be deployed in characterizing a possibility. In particular, an index α is used to characterize possibilities that take a stand on the existence of the possible individuals that α is used to keep track of.

Worlds are used to keep track of metaphysical possibilities. A **world** is a pair $\langle \delta, f \rangle$, where δ is an index and f is a function that assigns to each n-place atomic predicate a set of n-tuples of members of δ. Think of δ as the world's domain and of f as a specification of predicate-extensions at that world. Formally:

Definition 3 (Domains and extensions) *Let w be a world $\langle \delta, f \rangle$. Then:*

- *the **domain** of w is δ (in symbols: $\bar{w} = \delta$);*
- *the **extension** of P^n_k at w is $f(P^n_k)$ (in symbols: $[\![P^n_k]\!]_w = f(P^n_k)$).*

We may now characterize a notion of α-indiscernibility, where α is an index. The intuitive idea is that wolrds are α-indiscernible iff they are indistinguishable from the standpoint of someone who only has access to distinctions made available by the possible individuals that α is used to keep track of. Formally:

Definition 4 (α-indiscernibility) *For any worlds w, v and any index α, w and v are **α-indiscernible** (in symbols: $w \sim_\alpha v$) iff there is a permutation π of $\mathscr{D}$ such that:*[24]

- $(\alpha)^\pi = \alpha$

[24] I use the obvious notational conventions: $(A)^\pi = \{\pi(z) : z \in A\}$ $(A \subseteq \mathscr{D})$, $\langle a_1, \ldots, a_n \rangle^\pi = \langle \pi(a_1), \ldots, \pi(a_n) \rangle$ $(a_1, \ldots, a_n \in \mathscr{D})$, and $(A^n)^\pi = \{\langle a_1, \ldots, a_n \rangle^\pi : \langle a_1, \ldots, a_n \rangle \in A^n\}$ $(A^n \subseteq \underbrace{\mathscr{D} \times \cdots \times \mathscr{D}}_{n \text{ times}})$.

- $w^\pi = v$

  (*i.e.* $(\bar{w})^\pi = \bar{v}$ *and* $(\llbracket P^n_k \rrbracket_w)^\pi = \llbracket P^n_k \rrbracket_v$ *for each* P^n_k).

A **proposition** of is a set of worlds. We say that a proposition q is a **α-level proposition** if it never cuts across α-indiscriminable worlds:

$$w \in q \leftrightarrow v \in q, \quad \text{whenever } w \sim_\alpha v$$

Intuitively, an α-level proposition is a proposition that is available from the standpoint of someone who is able to make every distinction made available by the atomic predicates of L^H and the members of α. (Note that $\alpha \subseteq \beta$ entails that every α-level proposition is also a β-level proposition.)

C.3 Models and Algebras

For a given domain $\mathscr{D}$, we let a **model** (based on $\mathscr{D}$) be a pair $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ such that:

- $\mathcal{W}^{\mathscr{D}}$ is a set of worlds such that $\bigcup \left\{ \bar{w} : w \in \mathcal{W}^{\mathscr{D}} \right\} = \mathscr{D}$;

  The worlds $\mathcal{W}^{\mathscr{D}}$ are used to keep track of the ways for the world to be that the model counts as non-absurd, and therefore as possibilities.

- $@$ is a world in $\mathcal{W}^{\mathscr{D}}$;

  $@$ keeps track of the possibility that in fact obtains. Accordingly, for a proposition q to be **true** according to $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ is for it to be the case that $@ \in q$.

For $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ a model and α an index, we let the **algebra** $\mathfrak{A}^{\mathcal{W}^{\mathscr{D}}}_\alpha$ be the set of α-level propositions of $\mathcal{W}^{\mathscr{D}}$. Note that $\mathfrak{A}^{\mathcal{W}^{\mathscr{D}}}_\alpha$ is always a complete, atomic Boolean algebra, under the usual set theoretic operations. The more inclusive α is, the more fine grained the propositions in $\mathfrak{A}^{\mathcal{W}^{\mathscr{D}}}_\alpha$ are. So the $\mathfrak{A}^{\mathcal{W}^{\mathscr{D}}}_\alpha$ form a hierarchy of Boolean algebras, partially ordered by fineness of grain.

(*Note:* The fact that we allow for propositions $\mathfrak{A}^{\mathcal{W}^{\mathscr{D}}}_\alpha$ where α is not the domain of a world in $\mathcal{W}^{\mathscr{D}}$ is the key difference between my proposal and that of Stalnaker and Fritz.)

C.4 Truth for L^H

For a fixed model $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$:

Definition 5 (Variable Assignments) *Let o be an arbitrary object outside $\mathscr{D}$. (One can think of o as a "dummy" object, which is used to ensure that σ can be well-defined even if α is empty.)*

For $\alpha \subseteq \mathscr{D}$, we say that σ is an α-level variable assignment if:

- *for each x_i, $\sigma(x_i) \in \alpha \cup \{o\}$;*
- *for each xx_i, $\sigma(xx_i)$ is a non-empty subset of $\alpha \cup \{o\}$; and*
- *for each p_i, $\sigma(p_i) \in \mathfrak{A}_\alpha^{\mathcal{W}^{\mathscr{D}}}$.*

Note that $\alpha \subseteq \beta$ entails that every α-level variable assignment is also a β-level assignment.

Definition 6 (Notation)

1. *Let $a \dot{\subseteq} b$ be shorthand for $(a \subseteq b \wedge a \neq \emptyset)$.*
2. *For σ a α-level assignment, let $\sigma(x_{i_1}, \ldots, x_{i_n}, xx_{j_1}, \ldots, xx_{j_m})$ be shorthand for*
$$(\{\sigma(x_{i_1}), \ldots, \sigma(x_{i_k})\} \cup \sigma(xx_{j_1}) \cup \cdots \cup \sigma(xx_{j_l})) - \{o\}$$
where o is the "dummy" object.

Definition 7 (the proposition $[\![\phi]\!]^\sigma$) *For σ an α-level variable assignment and ϕ a formula of L^H, we let the proposition $[\![\phi]\!]^\sigma$ be defined as follows:*

- $[\![\bot]\!]^\sigma = \emptyset$
- $[\![x_i = x_j]\!]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : \sigma(x_i) \in \bar{w} \wedge \sigma(x_i) = \sigma(x_j)\right\}$
- $[\![P_k^n(x_{i_1}, \ldots, x_{i_n})]\!]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : \langle \sigma(x_{i_1}), \ldots, \sigma(x_{i_n})\rangle \in [\![P_k^n]\!]_w\right\}$
- $[\![x_i \prec xx_j]\!]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : \sigma(xx_j) \subseteq \bar{w} \wedge \sigma(x_i) \in \sigma(xx_j)\right\}$
- $[\![xx_i = xx_j]\!]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : \sigma(xx_i) \subseteq \bar{w} \wedge \sigma(xx_i) = \sigma(xx_j)\right\}$

- $[\![p_i]\!]^\sigma = \sigma(p_i)$
- $[\![\phi \equiv \psi]\!]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : [\![\phi]\!]^\sigma = [\![\psi]\!]^\sigma\right\}$
- $[\![\neg\phi]\!]^\sigma = \mathcal{W}^{\mathscr{D}} - [\![\phi]\!]^\sigma$
- $[\![\phi \wedge \psi]\!]^\sigma = [\![\phi]\!]^\sigma \cap [\![\psi]\!]^\sigma$
- $[\![\exists x_i\, \phi]\!]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : \exists z \in \bar{w}\left(w \in [\![\phi]\!]^{\sigma[x_i/z]}\right)\right\}$
- $[\![\exists xx_i\, \phi]\!]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : \exists a \dot{\subseteq} \bar{w}\left(w \in [\![\phi]\!]^{\sigma[xx_i/a]}\right)\right\}$
- $\left[\!\!\left[\underset{v_1,\ldots,v_k}{\exists p_i}\ \phi\right]\!\!\right]^\sigma = \left\{w \in \mathcal{W}^{\mathscr{D}} : \exists q \in \mathfrak{A}^{\mathcal{W}^{\mathscr{D}}}_{\sigma(v_1,\ldots,v_k)}\left(w \in [\![\phi]\!]^{\sigma[p_i/q]}\right)\right\}$

Whenever σ is an α-level assignment, $[\![\phi]\!]^\sigma$ is an α-level proposition (Theorem 1). So one can think of $[\![\phi]\!]^\sigma$ as the proposition expressed by ϕ (relative to σ) from the standpoint of someone who is in a position to make the distinctions that are made available by the atomic predicates of the language and by members of α.

When ϕ is a sentence, $[\![\phi]\!]^\sigma$ expresses an $\emptyset$-level proposition (Corollary 1), which does not depend on σ. So one can introduce the following:

Definition 8 *The **proposition expressed** by a sentence ϕ (in symbols: $[\![\phi]\!]$) is the $\emptyset$-level proposition $[\![\phi]\!]^\sigma$, for σ an arbitrary α-level assignment ($\alpha \subseteq \mathscr{D}$).*

Definition 9 (Truth) *With respect to a given model $\left\langle \mathcal{W}^{\mathscr{D}}, @\right\rangle$, a sentence ϕ of L^H is **true** iff $@ \in [\![\phi]\!]$.*

Definition 10 (Validity) *For $\mathcal{C}$ a class of models, a formula ϕ of L^H is **valid relative to $\mathcal{C}$** iff $@ \in [\![\phi]\!]^\sigma$ for any α-level assignment σ ($\alpha \in \mathscr{D}$). A formula ϕ of L^H is **valid** iff ϕ is valid relative to the class of all models.*

C.5 Basic Results[25]

Theorem 1 (Level Theorem) *For σ an α-level assignment,*

$$[\![\phi]\!]^\sigma \in \mathfrak{A}^{\mathcal{W}^{\mathscr{D}}}_\alpha$$

[25]Proofs are spelled out in a supplementary appendix: `web.mit.edu/arayo/www/oels-proofs.pdf`.

Corollary 1 (Base Level Corollary) *For ϕ a sentence and σ an α-level assignment,*

$$[\![\phi]\!]^\sigma \in \mathfrak{A}_\emptyset^{\mathcal{W}^\mathscr{D}}$$

As outlined by the next two definitions, a model $\left\langle \mathcal{W}^\mathscr{D}, @ \right\rangle$ can be used as a Kripke model (with a trivial accessibility relation) for the modal language $L^\Diamond$.

Definition 11 *For a given domain $\mathscr{D}$, a variable assignment (with no level specification) is a function that maps each type-e variable to an individual in $\mathscr{D}$ and each type-ee variable to a non-empty subset of $\mathscr{D}$.*

Definition 12 *For $\left\langle \mathcal{W}^\mathscr{D}, @ \right\rangle$ a model, σ a variable assignment, and ϕ a formula of $L^\Diamond$, we let $[\![\phi]\!]_K^\sigma$ be defined as follows:*

- $[\![x_i = x_j]\!]_K^\sigma = \left\{ w \in \mathcal{W}^\mathscr{D} : \sigma(x_i) \in \bar{w} \wedge \sigma(x_i) = \sigma(x_j) \right\}$
- $[\![P_k^n(x_{i_1}, \ldots, x_{i_n})]\!]_K^\sigma = \left\{ w \in \mathcal{W}^\mathscr{D} : \langle \sigma(x_{i_1}), \ldots, \sigma(x_{i_n}) \rangle \in [\![P_k^n]\!]_w \right\}$
- $[\![x_i \prec xx_j]\!]_K^\sigma = \left\{ w \in \mathcal{W}^\mathscr{D} : \sigma(xx_i) \subseteq \bar{w} \wedge \sigma(x_i) \in \sigma(xx_j) \right\}$
- $[\![xx_i = xx_j]\!]_K^\sigma = \left\{ w \in \mathcal{W}^\mathscr{D} : \sigma(xx_i) \subseteq \bar{w} \wedge \sigma(xx_i) = \sigma(xx_j) \right\}$
- $[\![\neg\phi]\!]_K^\sigma = \mathcal{W}^\mathscr{D} - [\![\phi]\!]_K^\sigma$
- $[\![\phi \wedge \psi]\!]_K^\sigma = [\![\phi]\!]_K^\sigma \cap [\![\psi]\!]_K^\sigma$
- $[\![\exists x_i\, \phi]\!]_K^\sigma = \left\{ w \in \mathcal{W}^\mathscr{D} : \exists z \in w \left(w \in [\![\phi]\!]_K^{\sigma[x_i/z]} \right) \right\}$
- $[\![\exists xx_i\, \phi]\!]_K^\sigma = \left\{ w \in \mathcal{W}^\mathscr{D} : \exists a \dot{\subseteq} w \left(w \in [\![\phi]\!]_K^{\sigma[xx_i/a]} \right) \right\}$
- $[\![\Diamond\psi]\!]_K^\sigma = \left\{ w \in \mathcal{W}^\mathscr{D} : \exists u \in \mathcal{W}^\mathscr{D} \, (u \in [\![\psi]\!]_K^\sigma) \right\}$

Theorem 2 (Kripkean Equivalence Theorem) *Let $\left\langle \mathcal{W}^\mathscr{D}, @ \right\rangle$ be a model, let ϕ be a formula of $L^\Diamond$, and let ϕ' be the corresponding formula of L^H. For any α-level assignment σ:*

$$[\![\phi]\!]_K^\sigma = [\![\phi']\!]^\sigma$$

Definition 13 *A formula ϕ of $L^\Diamond$ is Kripke-valid iff: for any model $\left\langle \mathcal{W}^\mathscr{D}, @ \right\rangle$ and any assignment σ, $@ \in [\![\phi]\!]_K^\sigma$.*

Corollary 2 (Validity Theorem) *Let $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ be a model, let ϕ be a sentence of $L^{\Diamond}$, and let ϕ' be the corresponding sentence of L^H. Then:*

$$\phi \textit{ is Kripke-valid iff } \phi' \textit{ is valid}$$

C.6 Indefinitely Extensible Models

Definition 14 (Extensions) *Let $w \in \mathcal{W}^{\mathscr{D}}$, let v be a variable of type e, and let $\phi(v)$ be formula of L^H in which at most v occurs free. We define the* ***extension*** *of $\phi(v)$ at w as follows:*

$$[\![\phi(v)]\!]^v_w = \left\{ z \in \bar{w} : w \in [\![\phi(v)]\!]^{\sigma[v/z]} \right\}$$

where σ is an arbitrary α-level assignment ($\alpha \subseteq \mathscr{D}$). (It is easy to verify that the definition doesn't depend on one's choice of σ.)

Definition 15 (Indefinitely Extensible Models) *Let v be a variable of type e and let $\phi(v)$ be a formula of L^H in which at most v occurs free. We say that $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ is* ***indefinitely extensible*** *with respect to $\phi(v)$ iff: for any $w \in \mathcal{W}^{\mathscr{D}}$ there is some $u \in \mathcal{W}^{\mathscr{D}}$ such that $|[\![\phi(v)]\!]^v_w| < |[\![\phi(v)]\!]^v_u|$. We say that $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ is* ***indefinitely extensible*** *(simpliciter) iff it is indefinitely extensible with respect to $x_1 = x_1$.*

Suppose L^H contains an atomic predicate I that expresses the concept of an island universe. Then a logical contingentist who accepts the argument for Open-Endedness in §3 should restrict attention to models that are indefinitely extensible with respect to I. For the argument of §3 purports to show that whenever there might have been some individuals xx, the xx can be used to specify a possibility according to which there are more island universes than there are xx. Notice, moreover, that a model that is indefinitely extensible with respect to I is always such as to satisfy (the sentence corresponding to) the first premise of Fritz's Puzzle:

(F1) $\Box\forall xx\,(I(xx) \rightarrow \Diamond\exists y(I(y) \wedge y \not\prec xx))$

And, of course, a model that is indefinitely extensible *simpliciter* is always such as to satisfy (the sentence corresponding to) the negation of the Necessitist Axiom.

Definition 16 *For some $k \in \mathbb{N}$, we shall say that a* ***pair model*** *is a model $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ at which the following sentences are both true at every world in $\mathcal{W}^{\mathscr{D}}$:*

$$\forall x_2 \forall x_3 \exists x_1 \left(P_k^3(x_1, x_2, x_3) \right)$$

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 \forall x_5 \forall x_6 \left(\left(P_k^3(x_1, x_2, x_3) \wedge P_k^3(x_4, x_5, x_6) \right) \rightarrow (x_1 = x_4 \leftrightarrow (x_2 = x_5 \wedge x_3 = x_6)) \right)$$

When attention is restricted to pair models, one can simulate ordered pairs, and therefore full second-order quantification (Boolos 1985). Accordingly, one can define a "strictly fewer than" predicate "$<$" such that

$$[\![xx_i < xx_j]\!]^\sigma \leftrightarrow |\sigma(xx_i)| < |\sigma(xx_j)|$$

Note that (the sentence corresponding to) $\Box \forall xx_1 \Diamond \exists xx_2 \, (xx_1 < xx_2)$ is true with respect to any indefinitely extensible model that is also a pair model.

Definition 17 $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ *is an* ***ideal model*** *iff $\left\{ \bar{w} : w \in \mathcal{W}^{\mathscr{D}} \right\}$ is an ideal on $\wp(\mathscr{D})$.*[26]

Ideal models are nice to work with because they deliver a Principle of Recombination of sorts, by ensuring that the individuals in any two worlds can be combined and that any subset of the individuals in a world could exist without the rest. (A stronger principle on which the individuals in any collection of worlds could be combined would only be true in models that are not indefinitely extensible, and is therefore not something that the logical contingentist would accept).

Proposition 1 *For any $\kappa > |\mathbb{N}|$, there is an ideal, pair, indefinitely extensible model $\left\langle \mathcal{W}^{\mathscr{D}}, @ \right\rangle$ with $|\mathscr{D}| \geq \kappa$.*

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[26]An ideal on $\wp(A)$ is a non-empty subset of $\wp(A)$ that is closed under subsets and finite unions.

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