Frege's Unofficial Arithmetic

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In *The Foundations of Arithmetic* and *The Basic Laws of Arithmetic*, Frege held the view that number-terms refer to objects.[1] Later in his life, however, he seems to have been open to other possibilities:

> Since a statement of number based on counting contains an assertion about a concept, in a logically perfect language a sentence used to make such a statement must contain two parts, first a sign for the concept about which the statement is made, and secondly a sign for a second-order concept. These second-order concepts form a series and there is a rule in accordance with which, if one of these concepts is given, we can specify the next. But still we do not have in them the numbers of arithmetic; we do not have objects, but concepts. How can we get from these concepts to the numbers of arithmetic in a way that cannot be faulted? Or are there simply no numbers in arithmetic? Could the numbers help to form signs for these second-order concepts, and yet not be signs in their own right?[2]

To illustrate Frege's point, let us consider the number-statement 'there are three cats'. It might be paraphrased in a first-order language as:[3]

(1) $(\exists_3 x)[\textsc{Cat}(x)]$.

If its logical form is to be taken at face value, (1) can be divided into two main logical components: first, the predicate '$\textsc{Cat}(\ldots)$', which for Frege refers to the (first-order) concept *cat*; and, second, the quantifier-expression '$(\exists_3 x)[\ldots(x)]$', which for Frege refers to a second-order concept (specifically, the second-order concept which is true of the first-order concepts under which precisely 3 objects fall).[4] Significantly, Frege would regard neither of these components as referring to an *object*.

Let us now consider a close cousin of 'there are three cats', namely, 'the number of the cats is three'. This sentence might be paraphrased as:

(2) the number of the cats $= 3$.

If its logical form is to be taken at face value, (2) cannot be divided into a predicate and a quantifier-expression, like (1). Instead, Frege would take 'the number of the cats' and '3' to be *names*, referring to numbers (which he regarded as objects).

Frege saw a deep connection between sentences like (1)—in which something is predicated of a *concept*—and sentences like (2)—in which something is predicated of the *number* associated with that concept. An effort to account for this connection was a main theme in his philosophy of arithmetic. But, after the discovery that Basic Law V leads to inconsistency, he found much reason for dissatisfaction with his original proposal. As evidenced by the quoted passage, he no longer felt confident about the possibility of getting from concepts to their numbers 'in a way that cannot be faulted'.

Towards the end of the passage, Frege considers an alternative: the view that there really are no numbers in arithmetic, and that—appearances to the contrary—numerals are not names of objects. They do not even instantiate a legitimate logical category, they are merely *orthographic* components of expressions standing for second-order concepts. The grammatical form of a sentence like (2) is therefore not indicative of its logical form. Presumably, 'the number of the cats = 3' is to be divided into two main logical components. First, the expression '…cats', which refers to the (first-order) concept *cat*; and, second, the expression 'the number of the … = 3', which refers to a second-order concept (specifically, the second-order concept which is true of the first-order concepts under which precisely 3 objects fall). The numeral '3' is merely an orthographic component of 'the number of the … = 3', in much the same way that 'cat' is an orthographic component of 'caterpillar'. The outermost logical form of (2) is therefore identical to that of (1). If, in addition, it turns out that the logical form of 'the number of the … = 3' corresponds to that of '$(\exists_3 x)[\ldots(x)]$', then the logical form of (1) is identical to that of (2).

It is unfortunate that Frege never spelled out his *unofficial* proposal (as we shall call it) in any detail. In particular, he said nothing about how first-order arithmetic might be understood. Luckily, Harold Hodes has developed and defended a version of the Unofficial Proposal.[5] On Hodes's reconstruction, a sentence '$\mathrm{F}(n)$' of the language of first-order arithmetic is to be regarded as abbreviating a higher-order sentence '$(\mathsf{F}X)((\exists_n x)[Xx])$', where '$(\exists_n x)[\ldots x]$' refers to a second-order concept, and '$(\mathsf{F}X)(\ldots X \ldots)$' refers to a *third*-order concept. For instance, the first-order sentence '$\textsc{Prime}(19)$' abbreviates a certain higher-order sentence '$(\mathsf{Prime}\,X)((\exists_{19} x)[Xx])$'.

On Hodes's version of the Unofficial Proposal, quantified sentences involve quantification over second-order concepts. More specifically, they involve quantification over *finite cardinality object-quantifiers*: the referents of quantifier-expressions of the form '$(\exists_n x)[\ldots x]$'.[6] Thus, the first-order '$\exists z \textsc{Prime}(z)$' would abbreviate the result of replacing the position occupied by '$(\exists_{19} x)[\ldots x]$' in '$(\mathsf{Prime}\,X)((\exists_{19} x)[Xx])$'

by a variable ranging over finite cardinality object-quantifiers, and binding the new variable with an initial existential quantifier. Hodes's account of first-order arithmetic therefore requires *third*-order quantification. And the obvious extension to nth-order arithmetic (for $n \geq 2$) would call for $(n+2)$th-order quantification. Such logical resources are increasingly problematic.[7]

Here we shall see that more modest resources will do. We will develop a version of the Unofficial Proposal within a *second*-order language, and show that it can be used to account for nth order arithmetic (for any finite n). This, in itself, is a surprising result. But it is especially important in light of the fact that, although the use of higher-order languages is often considered problematic, recent work has done much to assuage concerns about certain second-order resources.[8] We will also see that the Unofficial Proposal has important applications in the philosophy of mathematics.

1 A Transformation

We will see that there is a general method for 'nominalizing' arithmetical formulas as second-order formulas containing no mathematical vocabulary. As an example, consider 'The number of the cats is the number of the dogs'. This sentence might be nominalized as 'The cats are just as many as the dogs', or:

$$\hat{x}\ [\textsc{Cat}(x)] \approx \hat{x}\ [\textsc{Dog}(x)],$$[9]

where '$\approx$' expresses one-one correspondence.[10]

Consider now the sentence 'the number of the cats is 3'. It can be nominalized as:

$$3^f(\hat{x}\ [\textsc{Cat}(x)]);$$

where numeral-predicates are defined in the obvious way:

- $0^f(X) \ \equiv_{df}\ \forall v\ \neg X(v);$
- $1^f(X) \ \equiv_{df}\ \exists W \exists v\ (0^f(W)\ \wedge\ \neg W(v)\ \wedge\ \forall w\ (X(w) \leftrightarrow (W(w) \vee w = v)));$
- $2^f(X) \ \equiv_{df}\ \exists W \exists v\ (1^f(W)\ \wedge\ \neg W(v)\ \wedge\ \forall w\ (X(w) \leftrightarrow (W(w) \vee w = v)));$
- etc.

This sort of nominalization can easily be generalized. In order to do so, we work within a two-sorted second-order language L containing the following variables: first-order *arithmetical* variables, 'm_1', 'm_2', ..., monadic second-order *arithmetical* variables 'M_1', 'M_2', ..., first-order *general* variables, 'x_1', 'x_2',

..., and, for n a positive integer, n-place second-order *general* variables $\ulcorner X_1^n \urcorner$, $\ulcorner X_2^n \urcorner$, ...[11] We assume that L has been enriched with a single higher-level predicate '**N**' taking a monadic second-order general variable in its first argument-place and a first-order arithmetical variable in its second argument-place.[12] The well-formed formulas of L are defined in the usual way, with the proviso that an atomic formula can contain arithmetical variables only if it is of the form $\ulcorner m_i = m_j \urcorner$, $\ulcorner M_i m_j \urcorner$ or $\ulcorner \mathbf{N}(X_i^1, m_j) \urcorner$.[13]

On the intended interpretation, arithmetical variables are taken to range over the natural numbers, and general variables are taken to have an unrestricted range.[14] In addition, '$\mathbf{N}(X_i^1, m_j)$' is true just in case the number of the X_i^1s is m_j. Consider 'The number of the cats is three' as an example. It can be formalized in L as:

(3) $\exists m_1(\mathbf{N}(\hat{x}_1\ [\textsc{Cat}(x_1)],\ m_1)\ \wedge\ 3(m_1));$

where, again, the number predicates are defined in the obvious way:

- $0(m)\ \equiv_{df}\ \exists W(0^f(W) \wedge \mathbf{N}(W, m));$
- $1(m)\ \equiv_{df}\ \exists W(1^f(W) \wedge \mathbf{N}(W, m));$
- $2(m)\ \equiv_{df}\ \exists W(2^f(W) \wedge \mathbf{N}(W, m));$
- etc.[15]

Arithmetical predicates such as 'Successor', 'Sum' and 'Product' can easily be defined in terms of '**N**' and purely logical vocabulary.[16] So, without appealing to arithmetical primitives beyond '**N**', the whole of pure and applied second-order arithmetic can be expressed within L.

It will be convenient to introduce the following definitions, which are couched in purely logical vocabulary:

Definition 1 $\boldsymbol{F}(X)\ \equiv_{df}$

$$\neg\exists W(\exists w(\neg Ww \wedge \forall v(Xv \leftrightarrow (Wv \vee v = w))) \wedge W \approx X)$$

(there are at most finitely many Xs)

Definition 2 $\exists^f X\ \phi(X)\ \equiv_{df}$

$$\exists X\ (\boldsymbol{F}(X)\ \wedge\ \phi(X))$$

Our nominalization method can now be generalized to encompass the whole of first-order arithmetic by way of the following transformation:[17]

- $Tr(\ulcorner \exists m_i \ (\phi) \urcorner) = \ulcorner \exists^f Z_i \urcorner \frown Tr(\ulcorner \phi \urcorner)$;
- $Tr(\ulcorner m_i = m_j \urcorner) = \ulcorner Z_i \approx Z_j \urcorner$;
- $Tr(\ulcorner \mathbf{N}(X_i,\ m_j) \urcorner) = \ulcorner X_i \approx Z_j \urcorner$.

Intuitively, the transformation works by replacing talk of the *number* of the Fs by talk of the Fs themselves. As an example, let us return to 'the number of the cats is three'. It can be formalized in L as:

$$\exists m_1\ (\mathbf{N}(\hat{x}_1\ [\textsc{Cat}(x_1)],\ m_1)\ \wedge\ 3(m_1));$$

which Tr converts to:

$$\exists^f Z_1\ (\hat{x}_1\ [\textsc{Cat}(x_1)] \approx Z_1\ \wedge\ 3^f(Z_1));$$

or, equivalently:

$$3^f(\hat{x}_1\ [\textsc{Cat}(x_1)]).$$

For further illustration, note that 'the number of the cats is the number of the dogs' can be formalized in L as:

$$\exists m_1\ [\mathbf{N}(\hat{x}_1\ [\textsc{Cat}(x_1)],\ m_1) \wedge (\mathbf{N}(\hat{x}_1\ [\textsc{Dog}(x_1)],\ m_1)].$$

which Tr converts to:

$$\exists^f Z_1\ [\hat{x}_1\ [\textsc{Cat}(x_1)] \approx Z_1 \wedge \hat{x}_1\ [\textsc{Dog}(x_1)] \approx Z_1],$$

or, equivalently:

$$\hat{x}_1\ [\textsc{Cat}(x_1)] \approx \hat{x}_1\ [\textsc{Dog}(x_1)].$$

It is worth emphasizing that *mixed* identity statements such as '$m_i = x_j$' are not well-formed formulas of L, so our transformation has not been defined for them. Intuitively, this means that the transformation is undefined for sentences along the lines of 'The number 2 is Julius Caesar', which do not express internal properties of a mathematical structure. We call such sentences *Caesar sentences.*

This is as it should be. The view that numbers are objects led Frege to the uncomfortable question of whether the number belonging to the concept *cat* is, for instance, Julius Caesar. But in the context of our nominalizations, such questions never arise, because number-terms do not refer to objects. 'The number belonging to the concept *cat* is the number belonging to the concept *dog*' is nominalized as 'the objects falling under the concept *cat* are in one-one correspondence with the objects falling under the concept *dog*', and 'the number belonging to the concept *cat* is 3' is nominalized as 'there are three objects falling under the concept *cat*'.

The question whether Julius Caesar is the number belonging to the concept *cat* isn't only uncomfortable because it appears to be nonsensical. It also underscores a problem Paul Benacerraf made famous, that if mathematical terms refer to objects, then nothing in our mathematical practice determines *which* objects they refer to.[18] A remarkable feature of the Unofficial Proposal is that it avoids Benacerraf's Problem altogether. It would, however, be a mistake to conclude from this that the Unofficial Proposal is the last word on Benacerraf's Problem, since the inscrutability of reference pervades far beyond arithmetic.

2 Second-order Arithmetic

On the assumption that there are infinitely many objects in the range of the general variables of L, a certain kind of coding can be used extend Tr so that it encompasses *second-order* arithmetic (thanks here to ...). Intuitively, the coding works by representing each arithmetical concept M_i by a dyadic relation R_i. Specifically, we represent the fact that a number m_j falls under M_i by having it be the case that some concept W under which precisely m_j objects fall be such that some individual v bears R_i to all and only the individuals falling under W.[19]

We implement the coding by enriching our transformation with the following two clauses:[20]

- $Tr(\ulcorner \exists M_i \ (\phi) \urcorner) = \ulcorner \exists R_i \urcorner \frown Tr(\ulcorner \phi \urcorner)$;
- $Tr(\ulcorner M_i m_j \urcorner) = \ulcorner \exists v \ (\mathbf{F}(\hat{u}[R_i(v,u)]) \wedge Z_j \approx \hat{u}[R_i(v,u)]) \urcorner$.

3 Higher-order Arithmetic

It is possible to express any (non-Caesar) formula in the langauge of n-th order arithmetic as a formula of L for which Tr is defined, provided that the range of the general variables contains at least $\beth_{n-2}$ many objects.

Consider the case of third-order arithmetic. Intuitively, we proceed by pairing each second-order concept α_i with a triadic relation S_i in such a way that a set of numbers M_j falls under α_i just in case there is some object x with the following property:

(∗) For any number n, $M_j n$ holds just in case there is some object y such that there are exactly n vs satisfying $S_i(x, y, v)$.[21]

So that the 'empty' second-order concept (i.e. the second-order concept under which no first-order concept falls) may be represented, we let S_i represent the fact that M_j falls under α_i only if there is an object x such that it is *both* the case that (∗) is satisfied, and that there is no y such that $S_i(x, y, x)$. The 'empty' second-order concept can then be represented by any relation S_i such that for every x there is some y such that $S_i(x, y, x)$.

Formally, if 'α_i' is a monadic third-order variable restricted to the natural numbers,[22] we define a transformation C as follows:[23]

- $C(\ulcorner \exists \alpha_i \ \phi \urcorner) \ = \ \ulcorner \exists S_i \urcorner \frown C(\ulcorner \phi \urcorner)$

- $C(\ulcorner \alpha_i(M_j) \urcorner) \ =$
  $\ulcorner \exists x \ [\forall y(\neg S_i(x, y, x)) \wedge \forall m \ (M_j m \leftrightarrow \exists y(\mathbf{N}(\hat{v}[S_i(x, y, v)], \ m)))] \urcorner$

On the assumption that the range of the general variables contains least continuum many objects, it is easy to verify that, for any formula of third-order arithmetic, ϕ, on which C is defined, $\phi \leftrightarrow C(\phi)$.

By using n-adic relations instead of triadic ones, this procedure can be extended to n-th order arithmetic. And, on the assumption that the range of the general variables contains at least $\beth_{n-2}$ objects, it will be the case that, for any formula of n-th order arithmetic, ϕ, on which C is defined, $\phi \leftrightarrow C(\phi)$.

4 Numbering Numbers

One would like to be able to number cats. But one would also like to be able to number *numbers*. One would like to say, for example, that the number of primes smaller than ten is four. And, unfortunately, an expression such as '$\mathbf{N}(\hat{m}_i \ [\text{Prime-less-than-}10(m_i)], m_j)$' is not well-formed formula of L because '$\mathbf{N}$' can only admit of a general variable in its first argument-place.[24] To remedy the situation, we may define a predicate '$\mathbf{NN}(M_i, m_j)$', by appealing to the same sort of coding as before.

Informally, '$\mathbf{NN}(M_i, m_j)$' is to abbreviate a formula of L to the effect that there is a binary relation R with the following properties:

- For any number n, $M_i n$ holds just in case some member of the domain of R is paired with exactly $n+1$ objects;[25]
- every member of the domain of R is paired with finitely many objects;
- for any x and y in the domain of R, if the objects paired with x are as many as the objects paired with y, then $x = y$;
- the domain of R contains exactly m_j objects.[26]

The new predicate allows us to say that the number of primes smaller than ten is four. It also allows us to say that the number of primes smaller than three is the number of objects falling under the concept *cat*:

$$\exists m_2\ (\mathbf{NN}(\hat{m}_1\ [\textsc{Prime-less-than-6}(m_1)], m_2)\ \wedge\ \mathbf{N}(\hat{x}_1\ [\textsc{Cat}(x_1)], m_2)).$$[27]

And, as desired, our any expression of the form $\ulcorner \mathbf{NN}(M_i, m_j) \urcorner$ is definitionally equivalent to a well-formed formula of L.

5 Formulas of L and their Transformations

Our nominalization method is now complete.[28] Caesar sentences aside, any formula in the language of n-th order applied arithmetic can be expressed as a formula of L for which Tr is defined. And the result of applying Tr is always a formula with no mathematical vocabulary.

We may now give a general characterization of the relationship between a formula and its transformation. In order to do so, consider the following five principles, all of which hold on the intended interpretation of L:

1. $\forall X\ (\exists m(\mathbf{N}(X, m)) \rightarrow \exists! m(\mathbf{N}(X, m)))$

   (If m is a number of the Xs, then m is the number of the Xs.)

2. $\forall m \exists X\ \mathbf{N}(X, m)$

   (Given any number m, there are some objects such that m belongs to those objects.)

3. $\forall X\ (\exists m\ (\mathbf{N}(X, m)) \leftrightarrow \mathbf{F}(X))$

   (A number belongs to the Xs just in case the they are at most finite in number.)

4. $\forall X \forall Y\ [\forall m(\mathbf{N}(X, m) \rightarrow (Y, m)) \ \leftrightarrow \ X \approx \ Y)]$.

   (A number belonging to the Xs is also a number belonging to the Ys just in case the Xs are in one-one correspondence with the Ys.)

5. $\exists X\ \neg \mathbf{F}(X)$

   (There are infinitely many things in the range of the general variables)

Let $\mathcal{A}$ be the conjunction of these five principles, and let $\ulcorner \phi^{Tr} \urcorner$ be a notational variant for $Tr(\ulcorner \phi \urcorner)$. It is possible to show that, for any sentence ϕ of L,[29]

$$\mathcal{A} \vdash \ \phi \leftrightarrow \phi^{Tr}$$

where '$\vdash$' expresses derivability in a standard second-order deductive system. In order to prove this result, a few preliminaries are necessary.

Definition 3 $\boldsymbol{N}(R_i, M_j) \ \equiv_{df}$

$$\forall m(M_j m \leftrightarrow \exists v(\boldsymbol{N}(\hat{u}[R_i(v, u)], m))).$$

Definition 4 *If $m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}$ are arithmetical variables, we let*

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}}$$

abbreviate the following:

$$(\mathbf{N}(Z_{i_1}, m_{i_1}) \wedge \ldots \wedge \mathbf{N}(Z_{i_k}, m_{i_k}) \wedge \mathbf{N}(R_{j_1}, M_{j_1}) \wedge \ldots \wedge \mathbf{N}(R_{j_l}, M_{j_l})).$$

Definition 5 *If $\ulcorner \phi \urcorner$ is a formula of L, with free arithmetical variables $m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}$, we let $\ulcorner \phi \leftrightarrow^* \phi^{Tr} \urcorner$ abbreviate the universal closure of the following:*

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\phi \leftrightarrow \phi^{Tr}).$$

If $\ulcorner \phi \urcorner$ contains no free arithmetical variables, we let $\ulcorner \phi \leftrightarrow^* \phi^{Tr} \urcorner$ be $\ulcorner \phi \leftrightarrow \phi^{Tr} \urcorner$.

Finally, we proceed to our main result:

Theorem *If $\ulcorner \phi \urcorner$ is a well-formed formula of L, then $\mathcal{A} \vdash \phi \leftrightarrow^* \phi^{Tr}$.*

See appendix for proof. [An interesting feature of the proof is that the fifth conjunct of $\mathcal{A}$ is required only to ensure the adequacy of the coding for second-order variables set forth in section 2. In particular, the fifth conjunct is not required to prove a version of the theorem restricted to first-order arithmetic. On the other hand, without its fifth conjunct—or, alternatively, without a principle guaranteeing the existence of infinitely objects in the range of the arithmetical variables—the standard arithmetical axioms do not follow from $\mathcal{A}$.]

Corollary 1 (Completeness of $\mathcal{A}$ with respect to applied arithmetic.) *If $\ulcorner \phi \urcorner$ is a sentence of L and T is the set of true sentences of L which do not contain '**N**', then either $\mathcal{A} \cup T \vdash \phi$ or $\mathcal{A} \cup T \vdash \neg\phi$.*

Proof: Let $\ulcorner \phi \urcorner$ be a sentence of L. It is easy to verify that $\ulcorner \phi \urcorner^{Tr}$ does not contain '**N**'. Therefore, either $T \vdash \phi^{Tr}$ or $T \vdash \neg\phi^{Tr}$, since either $\phi^{Tr} \in T$ or $\neg\phi^{Tr} \in T$. But, since $\ulcorner \phi \urcorner$ contains no free variables, it follows from our Theorem that $\mathcal{A} \vdash \phi \leftrightarrow \phi^{Tr}$. So, either $\mathcal{A} \cup T \vdash \phi$ or $\mathcal{A} \cup T \vdash \neg\phi$. □

Corollary 2 *Suppose $\mathcal{A}$ holds when '$\boldsymbol{N}(X, m)$' is interpreted as 'the number of the Xs is m'. Let $\phi(m_i)$ be a well-formed formula of L, and let $\psi(Z_i)$ be $Tr(\phi(m_i))$. If there are at most finitely many Fs, then $\phi(m_i)$ is true of the number of the Fs just in case $\psi(Z_i)$ is true of the Fs.*[30]

Proof: Immediate from theorem.

6 Interpreting Second-Order Languages

We have taken care to ensure that the outputs of our transformation are always second-order formulas. So an interpretation for second-order quantifiers is all we need to make sense of our nominalizations. Frege took second-order quantifiers to range over *concepts*, but Fregean concepts might be considered problematic on the grounds that they constitute 'items' which are not objects.

Not any alternative will do. On Quine's interpretation, second-order logic is 'set-theory in sheep's clothing'. So we would have succeeded in eliminating number-terms from arithmetic only by making use of *set*-terms. And, from the perspective of the Unofficial Proposal, set-terms are presumably no less problematic than number-terms. Nor is any progress made by interpreting second-order logic as Boolos has suggested.[31] Some of our definitions make essential use of *polyadic* second-order quantifiers, which Boolos treats as ranging (plurally) over *ordered n-tuples*. And, again, from the perspective of the Unofficial Proposal, ordered-pair-terms are presumably no less problematic than numbers-terms.

Some deviousness is needed to avoid Fregean concepts without betraying the spirit of the Unofficial Proposal. One way of doing so is by defining second-order quantifiers implicitly, in terms of an *open-*

ended schema, as in McGee's 'Everything'. Another is by interpreting second-order logic as in Rayo and Yablo's 'Nominalism through De-Nominalization'. Alternatively, one might argue that genuine second-order quantification is to be accepted as a primitive.

7 Applications

Frege's Unofficial Proposal—the view that number-statements are to be eliminated in favor of their transformations—can take several different forms, depending on the sort of elimination one has in mind. On an approach like Hodes's, number-statements are taken to *abbreviate* their transformations. As a result, number-terms do not refer to objects, and there is room for rejecting the existence of numbers altogether. The Unofficial Proposal might therefore provide a basis for a nominalist philosophy of arithmetic.

It should be noted, however, that unless the universe is infinite, ϕ^{Tr} will not always have the truth-value that ϕ receives on its standard interpretation. In order to avoid infinity assumptions, a nominalist might claim that a number-statement ϕ abbreviates 'necessarily, $(\xi \rightarrow \phi^{Tr})$', where '$\xi$' is a sentence stating that there are infinitely many objects, such as '$\exists X\ \neg\mathbf{F}(X)$'. On the plausible condition that it is *possible* for the universe to infinite, 'necessarily, $(\xi \rightarrow \phi^{Tr})$' is true if and only if ϕ is true on its standard interpretation.[32]

A different approach towards the Unofficial Proposal might serve the purposes of the Neo-Fregean Program, championed by Bob Hale and Crispin Wright. Neo-Fregeans believe that Hume's Principle allows us to *reconceptualize* the state of affairs which is described by saying that the Fs are as many as the Gs, and that, on the reconceptualization, that same state of affairs is rightly described by saying that the *number* of the Fs is the *number* of the Gs.[33] A version of the Unofficial Proposal might allow Neo-Fregeans to make the more general claim that every number-statement ϕ describes—on the appropriate reconceptualization—the state of affairs which is otherwise described by ϕ^{Tr}.

Even if the Unofficial Proposal is to be abandoned altogether, it would be a mistake to neglect the connection between number-statements and their transformations described in section 5. For non-nominalist accounts of mathematics must yield the result that there is no special mystery about how one might come to know what the truth-values of mathematical sentences are. But, on the assumption that $\mathcal{A}$ can be known to be true, our theorem ensures that this goal can be achieved for the case of pure and applied arithmetic. Let ϕ be an arithmetical sentence of L. When $\mathcal{A}$ is known, it follows from our theorem that one is in a position to derive $\phi \leftrightarrow \phi^{Tr}$. So, insofar as one is in a position to know the truth of ϕ^{Tr}, which contains no arithmetical vocabulary, one is also in a position to know the truth of ϕ.[34] (Of course, one may not be in a position to know the truth of ϕ^{Tr}. In that case one is not, for all that has been said, in a position to know ϕ. But that cannot be used as an objection against a non-nominalist account of mathematical knowledge. Such an account is required to show that mathematical knowledge is no more mysterious than non-mathematical knowledge, not that all knowledge is unproblematic.)

8 Logicism

Our theorem provides us with a partial vindication of Logicism. For whenever ϕ is a sentence of pure arithmetic (appropriately expressed in L), ϕ^{Tr} is a sentence of pure second-order logic. Moreover, Tr allows us to express formulas of pure arithmetic as formulas of pure second-order logic in a way which preserves compositionality.[35] This would constitute a complete vindication of Logicism if it were true as a matter of pure logic that, for every appropriate ϕ, $Tr(\phi)$ has the truth-value that ϕ would receive on its standard interpretation. Unfortunately, the general equivalence in truth-value holds only if the universe is big enough, and the size of the universe is not a matter of pure logic. Tr doesn't reduce arithmetic to logic—but it comes close.

Appendix

The theorem is proved by induction on the complexity of $\ulcorner\phi\urcorner$. Trivial cases are omitted.

- Assume $\ulcorner\phi\urcorner = \ulcorner\mathbf{N}(X_i, m_j)\urcorner$. Then $\ulcorner\phi \leftrightarrow^* \phi^{Tr}\urcorner$ is the universal closure of

$$\mathbf{N}(Z_j, m_j) \rightarrow (\mathbf{N}(X_i, m_j) \leftrightarrow X_i \approx Z_j),$$

  which is an immediate consequence of $\mathcal{A}$ (first and fourth conjuncts).

- Assume $\ulcorner\phi\urcorner = \ulcorner m_i = m_j\urcorner$. Then $\ulcorner\phi \leftrightarrow^* \phi^{Tr}\urcorner$ is the universal closure of

$$(\mathbf{N}(Z_i, m_i)\ \wedge\ \mathbf{N}(Z_j, m_j)) \rightarrow (m_i = m_j \leftrightarrow Z_i \approx Z_j),$$

  which is an immediate consequence of $\mathcal{A}$ (first and fourth conjuncts).

- Assume $\ulcorner\phi\urcorner = \ulcorner M_j m_i\urcorner$. Then $\ulcorner\phi \leftrightarrow^* \phi^{Tr}\urcorner$ is the universal closure of

$$\overline{m_i, M_j} \rightarrow (M_j m_i \leftrightarrow \exists v\ (\mathbf{F}(\hat{u}[R_i(v, u)]) \wedge Z_j \approx \hat{u}[R_i(v, u)])).$$

  We make the following two assumptions:

$$\mathbf{N}(Z_i, m_i), \tag{1}$$

$$\mathbf{N}(R_j, M_j). \tag{2}$$

  Recall that (2) is shorthand for

$$\forall m(M_j m \leftrightarrow \exists v(\mathbf{N}(\hat{u}[R_i(v, u)], m))), \tag{3}$$

  from which it follows immediately that

$$(M_j m_i \leftrightarrow \exists v(\mathbf{N}(\hat{u}[R_i(v, u)], m_i))). \tag{4}$$

  From (1) and (4), together with $\mathcal{A}$ (first and fourth conjuncts), it follows that

$$M_j m_i \leftrightarrow \exists v(Z_j \approx \hat{u}[R_i(v,u)]). \tag{5}$$

And from (1) and (5), together with $\mathcal{A}$ (first, third and fourth conjuncts), it follows that

$$M_j m_i \leftrightarrow \exists v(\mathbf{F}(\hat{u}[R_i(v,u)]) \wedge Z_j \approx \hat{u}[R_i(v,u)]). \tag{6}$$

Discharging assumptions (1) and (2) we get:

$$\overline{m_i, M_j} \rightarrow (M_j m_i \leftrightarrow \exists v\ (\mathbf{F}(\hat{u}[R_i(v,u)]) \wedge Z_j \approx \hat{u}[R_i(v,u)])). \tag{7}$$

And the desired result follows from (7) by universal generalization.

- Assume $\ulcorner\phi\urcorner = \ulcorner\exists m_i\ \psi(m_i)\urcorner$. Let ψ have free arithmetical variables $m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}$ distinct from m_i.[36] Then $\ulcorner\phi \leftrightarrow^* \phi^{Tr}\urcorner$ is the universal closure of:

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\exists m_i\ \psi(m_i) \leftrightarrow \exists^f Z_i\ \psi^{Tr}(Z_i)).$$

By inductive hypothesis, the following is provable from HP:

$$\overline{m_i, m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\psi(m_i) \leftrightarrow \psi^{Tr}(Z_i)). \tag{1}$$

We make the following two assumptions:

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}}, \tag{2}$$

$$\exists m_i\ \psi(m_i). \tag{3}$$

By $\mathcal{A}$ (second and third conjuncts), it follows from (3) that

$$\exists m_i\ \exists W\ (\mathbf{F}(W)\ \wedge\ \mathbf{N}(W, m_i)\ \wedge\ \psi(m_i)). \tag{4}$$

So, by existential instantiation,

$$\mathbf{F}(C) \ \wedge \ \mathbf{N}(C,c) \ \wedge \ \psi(c). \tag{5}$$

But by (1) we have:

$$\overline{c, m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\psi(c) \leftrightarrow \psi^{Tr}(C)). \tag{6}$$

And from (2), (5) and (6) we may conclude

$$\psi^{Tr}(C). \tag{7}$$

Thus, making again use of (5),

$$\exists^f Z_i \psi^{Tr}(Z_i), \tag{8}$$

and, discharging assumption (3),

$$\exists m_i \ \psi(m_i) \rightarrow \exists^f Z_i \psi^{Tr}(Z_i). \tag{9}$$

Conversely, assume

$$\exists^f Z_i \ \psi^{Tr}(Z_i). \tag{10}$$

By existential instantiation:

$$\mathbf{F}(C) \ \wedge \ \psi^{Tr}(C). \tag{11}$$

It is a consequence of (10) and $\mathcal{A}$ (third conjunct) that

$$\exists m \mathbf{N}(C,m). \tag{12}$$

From (12) we obtain the following, by existential instantiation:

$$\mathbf{N}(C,c). \tag{13}$$

But by (1) we have:

$$\overline{c, m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\psi(c) \leftrightarrow \psi^{Tr}(C)). \tag{14}$$

And from (2), the second conjunct of (11), (13) and (14) we may conclude

$$\psi(c). \tag{15}$$

Thus,

$$\exists m_i \ \psi(m_i), \tag{16}$$

and, discharging assumption (10),

$$\exists^f Z_i \ \psi^{Tr}(Z_i) \rightarrow \exists m_i \ \psi(m_i). \tag{17}$$

Finally, we combine (9) and (17), and discharge assumption (2):

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\exists^n m_i \ \psi(m_i) \leftrightarrow \exists^f Z_i \psi^{Tr}(Z_i)). \tag{18}$$

The desired result is then obtained by universal generalization.

- Assume $\ulcorner \phi \urcorner = \ulcorner \exists M_j \ \psi(M_j) \urcorner$. Let ψ have free arithmetical variables $m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}$ distinct from M_j.[37] Then $\ulcorner \phi \leftrightarrow^* \phi^{Tr} \urcorner$ is the universal closure of:

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\exists M_j \ \psi(M_j) \leftrightarrow \exists R_j \ \psi^{Tr}(R_j)).$$

By inductive hypothesis, the following is provable from HP:

$$\overline{M_j, m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\psi(M_j) \leftrightarrow \psi^{Tr}(R_j)). \tag{1}$$

We make the following two assumptions:

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}}, \tag{2}$$

$$\exists M_j \ \psi(M_j). \tag{3}$$

By $\mathcal{A}$ (second, third and fifth conjuncts), it follows from (3) that

$$\exists M_j \ (\exists R(\mathbf{N}(R, M_j)) \ \wedge \ \psi(M_j)). \tag{4}$$

So, by existential instantiation,

$$\mathbf{N}(P, C) \ \wedge \ \psi(C). \tag{5}$$

But by (1) we have:

$$\overline{C, m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\psi(C) \leftrightarrow \psi^{Tr}(P)). \tag{6}$$

And from (2), (5) and (6) we may conclude

$$\psi^{Tr}(P). \tag{7}$$

Thus,

$$\exists R_j \ \psi^{Tr}(R_j), \tag{8}$$

and, discharging assumption (3),

$$\exists M_j \ \psi(M_j) \rightarrow \exists R_j \ \psi^{Tr}(R_j). \tag{9}$$

Conversely, assume

$$\exists R_j \ \psi^{Tr}(R_j). \tag{10}$$

By existential instantiation,

$$\psi^{Tr}(P). \tag{11}$$

The following is a logical truth:

$$\exists M \forall m (Mm \leftrightarrow \exists v(\mathbf{N}(\hat{u}[P(v,u)], m))). \tag{12}$$

But (12) is definitionally equivalent to

$$\exists M \ \mathbf{N}(P, M). \tag{13}$$

So, by existential instantiation,

$$\mathbf{N}(P, C). \tag{14}$$

But by (1) we have:

$$\overline{C, m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\psi(C) \leftrightarrow \psi^{Tr}(P)) \tag{15}$$

And from (2), (11), (14) and (15) we may conclude

$$\psi(C). \tag{16}$$

Thus,

$$\exists M_j \psi(M_j), \tag{17}$$

and, discharging assumption (10),

$$\exists R_j \psi^{Tr}(R_j) \rightarrow \exists M_j \ \psi(M_j). \tag{18}$$

Finally, we combine (9) and (18), and discharge assumption (2):

$$\overline{m_{i_1}, \ldots, m_{i_k}, M_{j_1}, \ldots, M_{j_l}} \rightarrow (\exists M_j \ \psi(M_j) \leftrightarrow \exists R_j \psi^{Tr}(R_j)). \tag{19}$$

The desired result is then obtained by universal generalization. □

Notes

[1] This is reflected in his definition of number. See, for instance Frege (1884) §67.

[2]*Notes for Ludwig Darmstaedter*, pp. 366-7. I have substituted 'second-order' for 'second-level'.

[3]As usual, '$(\exists_1 x)[\phi(x)]$' is defined as '$\exists x(\phi(x) \wedge \forall y(\phi(y) \rightarrow x = y))$', and (for $n > 1$) '$(\exists_n x)[\phi(x)]$' is defined as '$\exists x(\phi(x) \wedge (\exists_{n-1} y)[\phi(y) \wedge y \neq x])$'.

[4] For Frege, a first-order concept is a concept that takes objects as arguments, and an $(n+1)$th-order concept is a concept that takes nth-order concepts as arguments. See Frege (18931903), §21. Unless otherwise noted, we shall use 'concept' to mean '*first-order* concept'.

[5]See Hodes (1984). See also Wright (1983) pp. 36-40 and Bostock (1979), volume II chapter 1.

[6]See Hodes (1990) §3.

[7] Hodes (1990), observation 5, offers a nominalization of second-order arithmetic which does not exceed the resources of second-order logic. But it proceeds by encoding Ramsey sentences, and is therefore not a version of Frege's Unofficial Proposal.

[8] See Boolos (1984), Boolos (1985a), Boolos (1985b), McGee (2000) and Rayo and Yablo (2001).

[9]Syntactically, an expression of the form '$\hat{x}\ [\phi(x)]$' takes the place of a monadic second-order variable. But the result of substituting '$\hat{x}\ [\phi(x)]$' for 'Y' in a formula '$\Psi(Y)$' is to be understood as shorthand for:

$$\forall W\ (\forall x(W x \leftrightarrow \phi(x)) \rightarrow \Psi(W)).$$

[10]That is, '$X \approx Y$' abbreviates

$$\exists R\ [\forall w(Xw \rightarrow \exists! v(Yv \wedge Rwv)) \wedge \forall w(Yw \rightarrow \exists! v(Xv \wedge Rvw))]$$

[11]As a precaution against variable clashes, we divide monadic second-order general variables in two: the $\ulcorner X^1_{2i} \urcorner$—which we abbreviate $\ulcorner Z_i \urcorner$—will be paired with first-order arithmetical variables; the $\ulcorner X^1_{2i+1} \urcorner$—which we abbreviate $\ulcorner X_i \urcorner$—will be used for more general purposes. Also to avoid variable clashes, we divide dyadic second-order general variables in two: the $\ulcorner X^2_{2i} \urcorner$—which we abbreviate $\ulcorner R_i \urcorner$—will be

paired with second-order arithmetical variables; the ⌜X^2_{2i+1}⌝—which we abbreviate ⌜R^2_i⌝—will be used for more general purposes. Finally, we divide triadic second-order general variables in two: the ⌜X^3_{2i}⌝—which we abbreviate ⌜S_i⌝—will be paired with third-order arithmetical variables; the ⌜X^3_{2i+1}⌝—which we abbreviate ⌜R^3_i⌝—will be used for more general purposes. For $n > 3$, we use ⌜R^n_i⌝ as a terminological variant of ⌜X^n_i⌝. We will sometimes appeal to the introduction of unused variables. We employ 'm' as an unused first-order arithmetical variable, 'w', 'v' and 'u' as unused first-order general variables, 'M' as an unused second-order arithmetical variable, 'W', 'V' and 'U' as unused monadic second-order general variables, and, for each $n > 1$ (to be determined by context), we employ 'R' as an unused n-place second-order general variable. (It is worth noting that appeal to unused variables could be avoided by renumbering subscripts.) It will often be convenient regard 'x', 'y', and 'z' as arbitrary first-order general variables and 'X', 'Y' and 'Z' as arbitrary (monadic) second-order general variables.

[12] For a discussion of higher-order predicates see my ....

[13]Formally, the well-formed formulas of L can be characterized as follows: (a) ⌜$\mathbf{N}(X^1_i, m_j)$⌝ and ⌜$m_i = m_j$⌝ are formulas; (b) for any n-place atomic predicate ⌜P⌝ other than '**N**', ⌜$\mathrm{P}(x_{i_1}, \ldots, x_{i_n})$⌝ is a formula; ($c$) ⌜$M_i m_j$⌝ and ⌜$X^n_i(x_{j_i}, \ldots, x_{j_n})$⌝ are formulas; ($d$) if ⌜$\phi$⌝ and ⌜$\psi$⌝ are formulas, then ⌜$\neg\phi$⌝, ⌜$(\phi \wedge \psi)$⌝, ⌜$\exists m_i\ \phi$⌝, ⌜$\exists M_i\ \phi$⌝, ⌜$\exists x_i\ \phi$⌝ and ⌜$\exists X^n_i\ \phi$⌝ are formulas; and ($e$) nothing else is a formula.

[14]More precisely, *first-order* arithmetical variables are taken to range over the natural numbers, and *first-order* general variables are taken to have an unrestricted range. The range of the second-order variables is to be characterized accordingly. For instance, on a Fregean interpretation of second-order quantification, second-order arithmetical variables are taken to range over first-order concepts under which natural numbers fall, and second-order general variables are taken to range over first-order concepts under which arbitrary objects fall.

[15] We use number-predicates rather than numerals for the sake of simplicity, but it is worth noting that our nominalization could be carried out even if L was extended to contain numerals. To see this, note that—using standard techniques—any formula ϕ of the extended language can be transformed into an equivalent formula ϕ^* of the original language in which numerals have been eliminated in favor of corresponding number-predicates (defined as above). One can then identify the nominalization of ϕ with that of ϕ^*.

[16]The definitions run as follows:

- $\textsc{Successor}(m_i, m_j) \ \equiv_{df}$
  $\forall V \forall U \ [(\mathbf{N}(V, m_i) \wedge \mathbf{N}(U, m_j)) \rightarrow \exists u \ (Uu \wedge \hat{w} \ [Uw \wedge w \neq u] \approx \ V)];$

- $\textsc{Sum}(m_i, m_j, m_k) \ \equiv_{df}$
  $\forall V \forall U \forall W \ [(\mathbf{N}(V, m_i) \wedge \mathbf{N}(U, m_j) \wedge \mathbf{N}(W, m_k) \wedge \forall w \ (Vw \rightarrow \neg Uw)) \rightarrow \hat{w} \ [Vw \vee Uw] \approx \ W];$

- $\textsc{Product}(m_i, m_j, m_k) \ \equiv_{df}$
  $\forall V \forall U \forall W \ [(\mathbf{N}(V, m_i) \wedge \mathbf{N}(U, m_j) \wedge \mathbf{N}(W, m_k)) \rightarrow$
  $\exists R \ [\forall v \forall u ((Vv \wedge Uu) \rightarrow \exists! w (Ww \wedge Rvuw)) \wedge$
  $\forall w (Ww \rightarrow \exists! v \exists! u (Vv \wedge Uu \wedge Rvuw))]].$

[17]The remaining clauses are trivial:

- $Tr(\ulcorner \neg \phi \urcorner) = \text{`}\neg\text{'} \frown Tr(\ulcorner \phi \urcorner);$
- $Tr(\ulcorner \phi \wedge \psi \urcorner) = \text{`('} \frown Tr(\ulcorner \phi \urcorner) \frown \text{`}\wedge\text{'} \frown Tr(\ulcorner \psi \urcorner) \frown \text{`)'};$
- $Tr(\ulcorner \exists x_i \ (\phi) \urcorner) = \ulcorner \exists x_i \urcorner \frown (Tr(\ulcorner \phi \urcorner));$
- $Tr(\ulcorner \exists X_i \ (\phi) \urcorner) = \ulcorner \exists X_i \urcorner \frown (Tr(\ulcorner \phi) \urcorner);$
- $Tr(\ulcorner X_i x_j \urcorner) = \ulcorner X_i x_j \urcorner;$
- $Tr(\ulcorner \exists R^n_i \ (\phi) \urcorner) = \ulcorner \exists R^n_i \urcorner \frown (Tr(\ulcorner \phi) \urcorner);$
- $Tr(\ulcorner R^n_i (x_{j_1}, \ldots, x_{j_n}) \urcorner) = \ulcorner R^n_i (x_{j_1}, \ldots, x_{j_n}) \urcorner;$
- $Tr(\ulcorner x_i = x_j \urcorner) \ = \ \ulcorner x_i = x_j \urcorner \ ;$
- $Tr(\ulcorner \mathrm{P}^n_j (x_{i_1}, \ldots, x_{i_n}) \urcorner) = \ulcorner \mathrm{P}^n_j (x_{i_1}, \ldots, x_{i_n}) \urcorner.$

[18]See Benacerraf (1965).

[19]We represent the fact that the number *zero* falls under M_i by having it be the case that some object bears R_i to nothing. Thus, in order to represent the fact that zero does not fall under M_i we must have it be the case that every object bears R_i either to n objects for some $n > 0$ falling under M_i, or to infinitely many objects.

[20] Polyadic second-order quantification can be defined as monadic second-order quantification over sequences, which can be simulated within first-order arithmetic.

[21]We represent the fact that the number *zero* falls under M_j by having it be the case that some object y is such that there are no vs satisfying $S_i(x, y, v)$. Thus, in order to represent the fact that zero does not fall under M_j we must have it be the case that every object y is either such that that there are n vs satisfying $S_i(x, y, v)$ for some $n > 0$ falling under M_j, or such that there are infinitely many vs satisfying $S_i(x, y, v)$.

[22]For instance, on a Fregean interpretation of third-order quantification, 'α_i' ranges over second-order concepts under which fall first-order concepts under which fall natural numbers.

[23]The remaining clauses are trivial.

[24] In analogy with the above, we let the result of substituting '$\hat{m}_i$ $[\phi(m_i)]$' for 'M_j' in a formula '$\Psi(M_j)$' be shorthand for

$$\forall M\ (\forall m_i (M\, m_i \leftrightarrow \phi(m_i)) \rightarrow \Psi(M)).$$

[25]We require that a member of the domain of R be paired with $n + 1$ objects rather than n objects in order to accommodate the fact that the number zero might fall under M_i, since every member of the domain of R must be paired with at least one object.

[26]More precisely, '$\mathbf{NN}(M_i, m_j)$' is to abbreviate:

$$\exists R\ [\forall m_k\ (M_i m_k \leftrightarrow \exists w \exists W \exists u\ (\ Rwu \wedge \forall v\ (Wv \leftrightarrow (Rwv \wedge v \neq u)) \wedge \mathbf{N}(W, m_k))) \wedge$$

$$\forall w \forall v\ (Rwv \rightarrow \exists^f W \forall u\ (Wu \leftrightarrow Rwu)) \wedge$$

$$\forall w \forall v \forall W \forall V\ ((\exists u\ (Rwu) \wedge \forall u\ (Wu \leftrightarrow Rwu) \wedge \forall u\ (Vu \leftrightarrow Rvu) \wedge W \approx V) \rightarrow w = v) \wedge$$

$$\exists W\ (\forall v\ (Wv \leftrightarrow \exists u\ (Rvu)) \wedge \mathbf{N}(W, m_j))\,];$$

for m_k an unused variable.

[27]Whereas 'Cat(...)' may be regarded as an atomic predicate, 'Prime-less-than-6(...)' abbreviates a complex formula constructed using the arithmetical predicates defined in footnote 16.

[28]So far we have only been concerned with the arithmetic of finite cardinals. But it is worth noting that a similar transformation could be applied to the language of *infinite* cardinal arithmetic.

[29]Here and in what follows I assume that, as a precaution against variable clashes, ϕ contains no variables for the form $\ulcorner Z_i \urcorner$, $\ulcorner R_i \urcorner$ or $\ulcorner S_i \urcorner$.

[30] In fact, the result is slightly more general. Suppose $\phi(m_{i_1}, \ldots, m_{i_n})$ is a formula of L and let $\psi(Z_{i_1}, \ldots, Z_{i_n})$ be $Tr(\phi(m_{i_1}, \ldots, m_{i_n}))$; suppose, moreover, that there are at most finitely many F_1s, at most finitely many F_2s, ..., and at most finitely many F_ns. Then $\phi(m_{i_1}, \ldots, m_{i_n})$ is true when m_{i_1} is the number of the F_1s, m_{i_2} is the number of the F_2s, ..., and m_{i_n} is the number of the F_ns just in case $\psi(Z_{i_1}, \ldots, Z_{i_n})$ is true when the Z_{i_1}s are the F_1s, the Z_{i_2}s are the F_2s, ..., and the Z_{i_n}s are the F_ns.

[31] See Boolos (1984) and Boolos (1985a).

[32] For more on modal strategies, see part II of Burgess and Rosen (1997). Hodes discusses a modal strategy in section III of Hodes (1984).

[33] See Wright (1997), section I, and Hale (1997).

[34]For a more detailed treatment of this issue see my .... It is worth noting that the completeness of the second-order Dedekind-Peano axioms yields a similar result for the case of pure second-order arithmetic, and that the quasi-categoricity result in McGee (1997) yields a similar result for the case of pure set-theory.

[35]Unlike nominalization in terms of Ramsey sentences, Tr respects the logical connectives and quantifiers:

- $Tr(\ulcorner \neg\phi \urcorner) = \text{`}\neg\text{'} \frown Tr(\ulcorner \phi \urcorner)$,
- $Tr(\ulcorner \phi \wedge \psi \urcorner) = Tr(\ulcorner \phi \urcorner) \frown \text{`}\wedge\text{'} \frown Tr(\ulcorner \psi \urcorner)$,
- $Tr(\ulcorner \exists m_i \phi \urcorner) = \ulcorner \exists^f Z_i \urcorner \frown Tr(\ulcorner \phi \urcorner)$,
- $Tr(\ulcorner \exists M_i \phi \urcorner) = \ulcorner \exists R_i \urcorner \frown Tr(\ulcorner \phi \urcorner)$.

[36]The case where ψ has no free arithmetical variables distinct from m_i, and the case where ψ does not contain m_i free require trivial differences in terminology. We ignore them for the sake of brevity.

[37]The case where ψ has no free arithmetical variables distinct from M_j, and the case where ψ does not contain M_j free require trivial differences in terminology. We ignore them for the sake of brevity.

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