FREQUENCY INDEPENDENT FLEXIBLE SPHERICAL BEAMFORMING VIA RBF FITTING

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ABSTRACT

We describe a new method for sound analysis using a spherical microphone array without the use of quadrature over the sphere. Quadrature based solutions are very sensitive to the placement of microphones on the sphere, needing measurements to be made at exactly the quadrature positions. We propose to use fitting with band-limited radial basis functions (RBFs) rather than quadrature. Our approach results in frequency independent beamformer weights for flexibly placed microphone locations. Results are demonstrated using both synthetic and real spherical array data.

1. INTRODUCTION

Spherical arrays of microphones have previously been used for analysis of sound fields [1, 2, 3, 5, 4]. Such arrays allow creation of arbitrary beampatterns and the reconstruction of the sound field, both to a certain order in the eigenfunctions of the Helmholtz equation. Previous researchers have performed such analysis by using quadrature over the measurements on the sphere[2]. This requires exact placement of microphones at the quadrature points and using corresponding weights for quadrature. This may not be possible in an actual array due to the space required for mounting of microphones and wires and human error in placements. An alternate approach was proposed in [3] and was based on optimization of the quadrature weights for flexibly located microphones. However, this approach, while achieving very good and robust performance, lost the frequency independence of the beamformer weights.

Here we follow an alternate approach that still allows for robust solution of the flexible layout case but achieves it in a frequency independent manner. Our approach is based on the use of *band-limited radial basis function (RBF) interpolation* on the sphere. The use of band-limited functions is important for the error bounds and stability analysis of the spherical microphone array [6]. Results are demonstrated using both synthetic and real data.

2. RBF FITTING ON SPHERES

On the unit sphere (S_u) functions that depend only on the geodesic distance $\mu_{12} = \arccos(\mathbf{s_1} \cdot \mathbf{s_2})$ between points $\mathbf{s_1}$ and $\mathbf{s_2}$ can be expressed as

$$\phi(\mathbf{s_1}, \mathbf{s_2}) = \Phi(\mathbf{s_1} \cdot \mathbf{s_2}) = \Phi(\mu_{12}). \tag{1}$$

We term functions ϕ which are radial basis functions in the geodesic distance, as *distance functions on the sphere* (DFS). If $f(\mathbf{s_1} \cdot \mathbf{s_2}) = f(\mu) \in C^0[-1,1]$, then any such function can be written as a linear combination of Legendre polynomials

$$f(\mu) = \sum_{n=0}^{\infty} c_n P_n(\mu), \quad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(\mu) P_n(\mu) d\mu. \tag{2}$$

Using the addition theorem for spherical harmonics [7]

$$f(\mu) = \sum_{n=0}^{\infty} c_n P_n(\mathbf{s_1} \cdot \mathbf{s_2}) = \sum_{n=0}^{\infty} \frac{4\pi c_n}{2n+1} \sum_{m=-n}^{n} Y_n^{-m}(\mathbf{s_2}) Y_n^m(\mathbf{s_1}),$$

$$f(\mathbf{s_1} \cdot \mathbf{s_2}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} C_n^m(\mathbf{s_2}) Y_n^m(\mathbf{s_1}) \quad C_n^m(\mathbf{s_2}) = \frac{4\pi c_n Y_n^{-m}(\mathbf{s_2})}{2n+1}.$$

Surface δ function: An example of a DFS is the surface δ function, which has already been considered for use in beamforming [5, 3]. This function satisfies

$$\int_{S_u} f(\mathbf{s_1}) \delta(\mathbf{s_1}; \mathbf{s_2}) dS(\mathbf{s_1}) = f(\mathbf{s_2}), \quad \int_{S_u} \delta(\mathbf{s_1}; \mathbf{s_2}) dS(\mathbf{s_1}) = 1.$$

Since $f(\mathbf{s_1}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_n^m Y_n^m(\mathbf{s_1})$ and because the Y_n^m are orthonormal on the unit sphere, using the addition theorem

$$\delta(\mathbf{s_1}; \mathbf{s_2}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Y_n^{-m}(\mathbf{s_1}) Y_n^m(\mathbf{s_2}) = \sum_{n=0}^{\infty} \frac{(2n+1)P_n(\mathbf{s_1} \cdot \mathbf{s_2})}{4\pi}.$$

Band-limited surface δ function: We can write a band-limited version of the spherical δ function, since we will require our approximations to satisfy a bandwidth constraint. This function will have the properties of the δ function when applied to band-limited functions on the sphere. A function with a band limit p on the sphere can be written as

$$f(\mathbf{s_1}) = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} f_n^m Y_n^m. \tag{3}$$

Clearly, the desired properties of the δ function are satisfied by

$$\delta^{(p)}(\mathbf{s_1} \cdot \mathbf{s_2}) = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} Y_n^{-m}(\mathbf{s_1}) Y_n^m(\mathbf{s_2}) = \sum_{n=0}^{p-1} \frac{(2n+1)P_n(\mathbf{s_1} \cdot \mathbf{s_2})}{4\pi}.$$

Using the addition theorem, we can explicitly rewrite this as

$$\begin{aligned} \delta^{(p)}(\mathbf{s_1} \cdot \mathbf{s_2}) &= \frac{p}{4\pi} \frac{P_{p-1}(\mathbf{s_1} \cdot \mathbf{s_2}) - P_p(\mathbf{s_1} \cdot \mathbf{s_2})}{1 - (\mathbf{s_1} \cdot \mathbf{s_2})} \\ \delta^{(p)}(\mathbf{s_0} \cdot \mathbf{s_0}) &= \frac{p^2}{4\pi} \end{aligned} \tag{4}$$

2.1. Fitting with Distance Functions on the Sphere

DFS functions can be used as a basis for continuous surface functions on the sphere. If a surface function $\Phi(\mathbf{s_1})$ is evaluated at N points, we can fit it by solving the linear system.

$$\Phi(\mathbf{s_q}) = \sum_{j=1}^{N} a_j f_j(\mathbf{s_q}) \qquad f_j(\mathbf{s}) = f(\mathbf{s_j} \cdot \mathbf{s}) \tag{5}$$

This is a linear system with N unknowns and N equations and can be solved to find the a_j. In general, since we fit with band-limited functions to ensure the band limit condition on Φ, we will have $N \geq p^2$. We can regularize this fit by standard methods, e.g., Tikhonov regularization, to account for noise in our data.

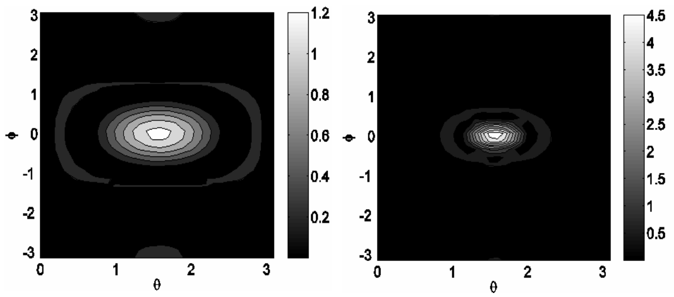

Fig. 1. A plot of the surface $\delta^{(p)}$ function for $p = 4$ and $p = 8$. We develop a theory for interpolation of scattered data on the the sphere using these band-limited functions of the geodesic distance on the sphere.

Equivalence of DFS and spherical harmonic expansions: Let a surface function Φ be written as

$$\Phi(\mathbf{s}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_n^m Y_n^m(\mathbf{s}) \tag{6}$$

By the Funk-Hecke Theorem (see e.g., [7]) we can rewrite this sum as,

$$\Phi(\mathbf{s}) = \int_{S_u} f(\mathbf{s_1} \cdot \mathbf{s}) \Psi(\mathbf{s_1}) ds(\mathbf{s_1}) \tag{7}$$

where $f(\mu) = f(\mathbf{s_1} \cdot \mathbf{s})$ is a DFS with $f(\mu) \in C^0[-1, 1]$, and $\Psi(\mathbf{s_1})$ is another surface function. Using the orthonormality of Y_n^m we find

$$\begin{aligned} C_n^m &= \int_{S_u} \Phi(\mathbf{s}) Y_n^{-m}(\mathbf{s}) dS(\mathbf{s}) \\ &= \int_{S_u} \int_{S_u} f(\mathbf{s_1} \cdot \mathbf{s}) \Psi(\mathbf{s_1}) dS(\mathbf{s_1}) Y_n^{-m}(\mathbf{s}) dS(\mathbf{s}) \\ &= \int_{S_u} \Psi(\mathbf{s_1}) dS(\mathbf{s_1}) \int_{S_u} f(\mathbf{s_1} \cdot \mathbf{s}) Y_n^{-m}(\mathbf{s}) dS(\mathbf{s}) \\ &= \lambda_n \int_{S_u} \Psi(\mathbf{s_1}) Y_n^{-m}(\mathbf{s_1}) dS(\mathbf{s_1}) \end{aligned}$$

where

$$\lambda_n = 2\pi \int_{-1}^{1} f(\mu) P_n(\mu) d\mu. \tag{8}$$

If we consider the DFS expansion of $\Phi(\mathbf{s})$ as

$$\Phi(\mathbf{s}) = \sum_{j=1}^{\infty} a_j f_j(\mathbf{s}),$$

we can relate the coefficients of the expansion a_j to the C_n^m as follows

$$\begin{aligned} C_n^m &= \int_{S_u} \Phi(\mathbf{s}) Y_n^{-m}(\mathbf{s}) dS(\mathbf{s}) \qquad (9) \\ &= \sum_{j=1}^{\infty} a_j \int_{S_u} f(\mathbf{s_j} \cdot \mathbf{s}) Y_n^{-m}(\mathbf{s}) dS(\mathbf{s}) = \lambda_n \sum_{j=1}^{\infty} a_j Y_n^{-m}(\mathbf{s_j}) \end{aligned}$$

In this way we can relate the Spherical Harmonic and DFS expansions of an arbitrary surface function. This shows that these two bases can be used interchangeably.

Expansion with basis of δ functions: Using Eq. (9), we can write

$$\Phi(\mathbf{s}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_n^m Y_n^m(\mathbf{s}), \; C_n^m = \lambda_n \sum_{j=1}^{N} a_j Y_n^{-m}.$$

If we set $f(\mathbf{s_j} \cdot \mathbf{s}) = f(\mu) = \delta^{(p)}(\mathbf{s_j}, \mathbf{s})$, the equation for λ_n becomes

$$\lambda_n = 2\pi \int_{-1}^{1} \frac{1}{4\pi} \sum_{n'=0}^{p-1} (2n'+1) P_{n'}(\mu) P_n(\mu) d\mu = \begin{cases} 1 & n < p \\ 0 & n \geq p \end{cases}$$

Therefore,

$$C_n^m = \sum_{j=1}^{N} a_j Y_n^{-m}(\mathbf{s_j}) \qquad N = p^2 \tag{10}$$

This will give the exact values for C_n^m for $n < p$. Therefore, this is an exact fit for functions of band-limit p.

2.2. Quadrature using truncated δ function fitting

Since for a band limited function $\Phi^{(p)}(\mathbf{s})$ we can write that

$$\Phi^{(p)}(\mathbf{s}) = \sum_{j=1}^{p^2} a_j \delta_j^{(p)}(\mathbf{s}),$$

we can calculate the quadrature over the sphere as follows

$$\int_{S_u} \Phi^{(p)}(\mathbf{s}) dS(\mathbf{s}) = \sum_{j=1}^{p^2} a_j \int_{S_u} \delta^{(p)}(\mathbf{s} \cdot \mathbf{s_j}) dS(\mathbf{s}) = \sum_{j=1}^{p^2} a_j. \tag{11}$$

This gives exact quadrature for band-limited functions of band less than p.

3. SOUND FIELD REPRESENTATION

When sound is captured by a sound-hard spherical microphone array, the array scatters the incident sound field. Therefore, the sound measured is a combination of the incident and scattered fields,

$$\psi(\mathbf{r}) = \psi_{in}(\mathbf{r}) + \psi_{scat}(\mathbf{r}). \tag{12}$$

Since the incoming sound ψ_{in} satisfies the Helmholtz equation, two representations for this sound field are

$$\psi_{in}(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_n^m R_n^m(\mathbf{r}) = \frac{1}{4\pi} \int_{S_u} e^{ik\mathbf{s} \cdot \mathbf{r}} \mu_{in}(\mathbf{s}) dS(\mathbf{s}), \tag{13}$$

where the R_n^m are the elementary spherical regular solutions of the Helmholtz equation and μ_{in} is the complex amplitude characterizing the magnitude and phase of the plane wave in the direction $\mathbf{s}$. These two expressions are related due to the Gegenbauer expansion of the plane wave [8]

$$\begin{aligned} e^{ik\mathbf{s} \cdot \mathbf{r}} &= 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n Y_n^{-m}(\mathbf{s}) R_n^m(\mathbf{r}), \qquad (14) \\ R_n^m(\mathbf{r}) &= \frac{i^{-n}}{4\pi} \int_{S_u} e^{ik\mathbf{s} \cdot \mathbf{r}} Y_n^m(\mathbf{s}) dS(\mathbf{s}). \end{aligned}$$

3.1. Sound analysis with a spherical array

In the classical approach to beamforming, the signals recorded at the microphones in the array are weighted and combined to achieve a beampattern. In the original spherical beamformer [2], the weight that must be applied to a microphone at a measurement point on the array is computed from the scattering solution for a plane-wave off a sphere. The beamformer shape in a particular look direction is expressed in terms of the spherical harmonics. Then, using the fact that the measured sound can be expanded in spherical harmonics, weights to achieve a beamformer shape are derived for the measured data [3].

An alternate approach was suggested in [5], where a plane-wave decomposition was used to represent the incoming sound-field. In this approach, plane-waves of a given order p (the plane wave form obtained by restricting the outer sum in Eq. (14) to p terms) from a set of directions were identified. This set of directions can be chosen independently from the directions of the microphones. Then, the strength of the plane wave from a given direction was identified as the strength of the sound from that direction.

While the beamformer approach can be followed using the fit function, since out interest is in playback of binaural sound using the sound recorded at a spherical array following the approach presented in [6], we will consider the plane-wave expansions.

In [6] an explicit expression for the complex amplitude μ_{in} was derived

$$\mu_{in}(\mathbf{s}') = \sum_{n=0}^{p-1}\sum_{m=-n}^{n} i^{-n} A_n^m Y_n^m(\mathbf{s}') = -i(ka)^2 \sum_{n=0}^{p-1} i^{-n} h_n'(ka) \times$$
$$\times \int_{S_u} \psi(\mathbf{s}) \sum_{m=-n}^{n} Y_n^{-m}(\mathbf{s}) Y_n^m(\mathbf{s}') dS(\mathbf{s}),$$

where $h_n(ka)$ are the spherical Hankel functions of the first kind and the prime in h_n' denotes a derivative with respect to the argument; $\mathbf{s}'$ is the direction of the plane-wave for which the intensity is sought, $\mathbf{s}$ is the variable of integration over the surface of the sphere, and a is the radius of the sphere. This allows the calculation of μ_{in} using numerical quadrature over the sphere. We will provide an alternate method for its evaluation using a fitting in terms of DFS here.

3.2. Choice of band-limit

As discussed in [6], the band limit p chosen to represent the plane waves should be adapted according to the frequency, so that the bandwidth is high enough that plane-waves of a certain frequency can be represented spatially on the surface of the sphere but not too high so that high frequency noise is not amplified by the fitting procedure. The error in truncating the Gegenbauer expansion of a single plane wave input field at p terms is

$$\epsilon_p(\mathbf{s},\mathbf{r}) = e^{ik\mathbf{s}\cdot\mathbf{r}} - 4\pi \sum_{n=0}^{p-1}\sum_{m=-n}^{n} i^n Y_n^{-m}(\mathbf{s}) R_n^m(\mathbf{r}) \quad (15)$$
$$= \sum_{n=p}^{\infty} (2n+1) i^n j_n(kr) P_n\left(\frac{\mathbf{r}\cdot\mathbf{s}}{r}\right)$$

Assume that the domain of interest can be enclosed inside a sphere of radius R, the following error bound was established in [4]

$$|\epsilon_p(\mathbf{s},\mathbf{r})| = \left|\sum_{n=p}^{\infty} (2n+1) i^n j_n(kr) P_n\left(\frac{\mathbf{r}\cdot\mathbf{s}}{r}\right)\right| \quad (16)$$
$$\leqslant \frac{2}{p!}\left(\frac{kR}{2}\right)^{p+1} \exp\left(\frac{kR}{2}\right) = \delta_p, \quad p \geqslant 1.$$

For relatively low ($kR < 1$) or moderate ($kR \approx 1$) frequencies, Eq. (16) provides relatively low p (e.g. for $kR = 2$ we have $|\epsilon_p| < 2e/p!$). For higher frequencies ($kR \gg 1$) asymptotic analysis (e.g., [7]) shows that p should be always larger than kR and

$$|\epsilon_p(\mathbf{s},\mathbf{r})| \lesssim \exp\left\{-\frac{1}{3}\left[2\frac{p-kR}{(kR)^{1/3}}\right]^{3/2}\right\} = \delta_p, \quad kR \gg 1. \quad (17)$$

As $\epsilon_p(\mathbf{s},\mathbf{r})$ can be uniformly bounded and the incident field can be represented as a superposition of plane waves, we can obtain from the overall error of approximation of the incident field by the band-limited function $\psi_{in}^{(p)}(\mathbf{r})$ inside a sphere of radius R:

$$\left|\psi_{in}(\mathbf{r}) - \psi_{in}^{(p)}(\mathbf{r})\right| \leqslant \frac{1}{4\pi}\int_{S_u} |\epsilon_p(\mathbf{s},\mathbf{r})| \, |\mu_{in}(\mathbf{s})| \, dS(\mathbf{s}) \quad (18)$$
$$\leqslant \max|\epsilon_p(\mathbf{s},\mathbf{r})| \max|\mu_{in}(\mathbf{s})| \lesssim \delta_p \max|\mu_{in}(\mathbf{s})| = \epsilon_s,$$

where δ_p can be selected according to Eq. (16) or Eq. (17). The latter formula can be inverted to determine p based on the specified accuracy ϵ_s. E.g., for $\epsilon_p = 0.02$ we have:

$$p \approx kR + \frac{1}{2}\left(3\ln\frac{\max|\mu_{in}(\mathbf{s})|}{\epsilon_s}\right)^{2/3} (kR)^{1/3}, \quad kR \gg 1. \quad (19)$$

In multifrequency analysis, we increase p along with the frequency as guided by Eq. (19) to avoid numerical errors.

4. CALCULATING PLANE WAVE AMPLITUDES

Here, we seek to obtain these strengths μ_{in} for particular sets of look directions $\mathbf{s}'$.We apply the truncated δ function fitting approach to quadrature shown in section 2.2 to find a new way of calculating the μ_{in}. This gives an exact expression for μ_{in} in terms of the a_j (the fitting coefficients)

$$\mu_{in}(\mathbf{s}') = -i(ka)^2 \sum_{n=0}^{p-1} i^{-n} h_n'(ka) \int_{S_u} \sum_{j=1}^{N} a_j \delta_j(\mathbf{s}) \times$$
$$\times \left(\sum_{m=-n}^{n} Y_n^{-m}(\mathbf{s}) Y_n^m(\mathbf{s}')\right) dS(\mathbf{s})$$

Simplifying this expression and using the orthonormality of the Spherical Harmonics, we get,

$$\begin{aligned}\mu_{in}(\mathbf{s}') &= -i(ka)^2 \sum_{j=1}^{N} a_j \sum_{n=0}^{p-1} i^{-n} h_n'(ka) \sum_{m=-n}^{n} Y_n^{-m}(\mathbf{s_j}) Y_n^m(\mathbf{s}') \\ &= -i(ka)^2 \sum_{j=1}^{N} a_j \sum_{n=0}^{p-1} i^{-n} h_n'(ka) P_n(\mathbf{s_j}\cdot\mathbf{s}')\left(\frac{2n+1}{4\pi}\right) \quad (20)\end{aligned}$$

This expression allows us to calculate μ_{in} by direct summation without having to rely on traditional spherical quadrature.

5. RESULTS

Simulations: We first tested our analysis approach using simulated sound generated according to the solution to the problem of sound scattering off of a sound-hard sphere [7].

$$\psi_S(\mathbf{s};\mathbf{s}') = \frac{i}{(ka)^2} \sum_{n=0}^{p-1} \frac{i^n (2n+1) P_n(\mathbf{s}\cdot\mathbf{s}')}{h_n'(ka)} \quad (21)$$

This gives a way to calculate the measured sound at each microphone location up to bandwidth p. Using this formula we calculated the noise free sound that would be measured on the sphere with vocal and music sound coming from directions (θ, φ) $\mathbf{s}_{voice} = (.86161, 1.8025)$ and $s_{music} = (.93373, -.55698)$. These values were chosen to match the directions of the sound used in the real world experiment described later. Also, to make the simulation realistic we used, as the locations of the microphones on the sphere, the actual locations of the microphones on the physical array. The microphones were located at approximately the Fliege points [9] with the bottom 4 points missing and also 2 more points missing to account for broken microphones.

Using our approach, we compute the complex amplitude of the waves μ_{in} in every direction in a uniform grid of points on the surface of the sphere (we use a grid of 32×32 points in θ and φ).

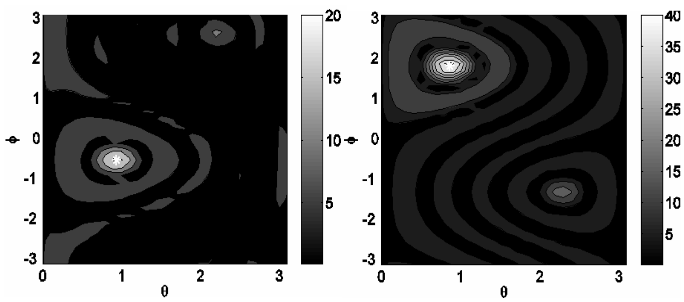

Fig. 2. The plane-wave coefficients for a synthetic case with (left) a music source at $(\theta, \varphi) = (0.93373, -0.55698)$, and (right) a voice source at $(\theta, \varphi) = (.86161, 1.8025)$. In each case the source (indicated with a *) is captured perfectly.

Clearly in both cases the peak of the μ_{in} value occurs at the actual location of the sound source.

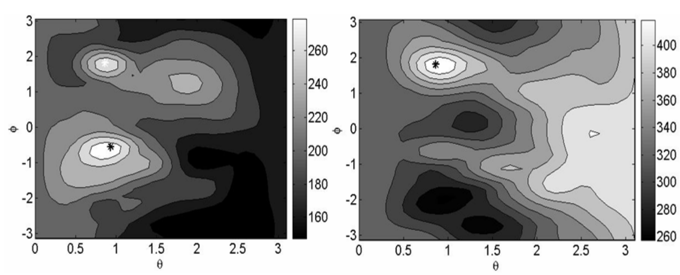

Fig. 3. A plot of reconstructed plane-wave strengths for two frames of a scene with a voice and a music source. Real source locations are shown with marks. The right picture shows a frame where only one source was active.

Experimental Results: We also used our beamformer on real data gathered by a spherical microphone [3]. The real world consisted of 2 sound sources, at the same locations as in the simulation. One was a vocal and one a music source. In the real world measurements we also have to deal with noise and reflections of the sound off of the walls of the room in which the measurement was made. Performing our beamforming on the measured sound, we got the values for μ_{in} for the two sounds. Fig. 3 shows the value at a given frequency for a time-frame in which the source was active. Both of these plots clearly show a peak in the values of μ_{in} at the sources of the sounds. There are multiple smaller peaks due to the noise present from other sources, but we are clearly able to identify the source direction of the primary sound.

6. CONCLUSION

In this paper we have presented a new way to use a spherical microphone array to perform beamforming on measured sound. We allowed for flexibility in microphone placement by using quadrature based on fitting with DFS functions. This approach provides an alternate methodology to previous work in the area. Our simulation and results clearly show that our beamforming works even when 6 of the microphones are missing. In standard quadrature based approaches the lack of 6 quadrature points would have introduced large error in the quadrature. Thus we are able to allow for flexibility in the placement of the microphones in the spherical array. Our only restriction is that we have enough measurement points to solve the linear system necessary to calculate the fitting. For this we need number of microphones greater than p^2 for the band limit p.

After we completed this work, we came across the recent theoretical study [10], where a similar fitting based approach is suggested theoretically, and implemented for 1-D arrays. However, in that work practical details of how the fitting can be done respecting bandwidth limits and arbitrary distributions of microphones on the sphere are not presented.

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