15.401 Finance Theory I

Alex Stomper
MIT Sloan School of Management
Institute for Advanced Studies, Vienna

Lecture 6: Options

_ Introduction to options

_ Option payoffs

_ Corporate securities as options

_ Use of options

_ Basic properties of options

_ Binomial Option Pricing Model

_ Black-Scholes option pricing formula

Readings:

_ Brealey, Myers and Allen, Chapters 21 - 22

_ Bodie, Kane and Markus, Chapters 20 - 21

Option types:

Call: The right to buy an asset (the underlying asset) for a given price (exercise price) on or before a given date (expiration date)

Put: The right to sell an asset for a given price on or before the expiration date

Exercise styles:

European: Owner can exercise the option only on expiration date

American: Owner can exercise the option on or before expiration date

Key elements in defining an option:

- Underlying asset and its price S
- Exercise price (strike price) K
- Expiration date (maturity date) T (today is 0)
- European or American

Example. A European call option on IBM with exercise price $100. It gives the owner (buyer) of the option the right (not the obligation) to buy one share of IBM at $100 on the expiration date. Depending on the share price of IBM on the expiration date, the option's payoff is:

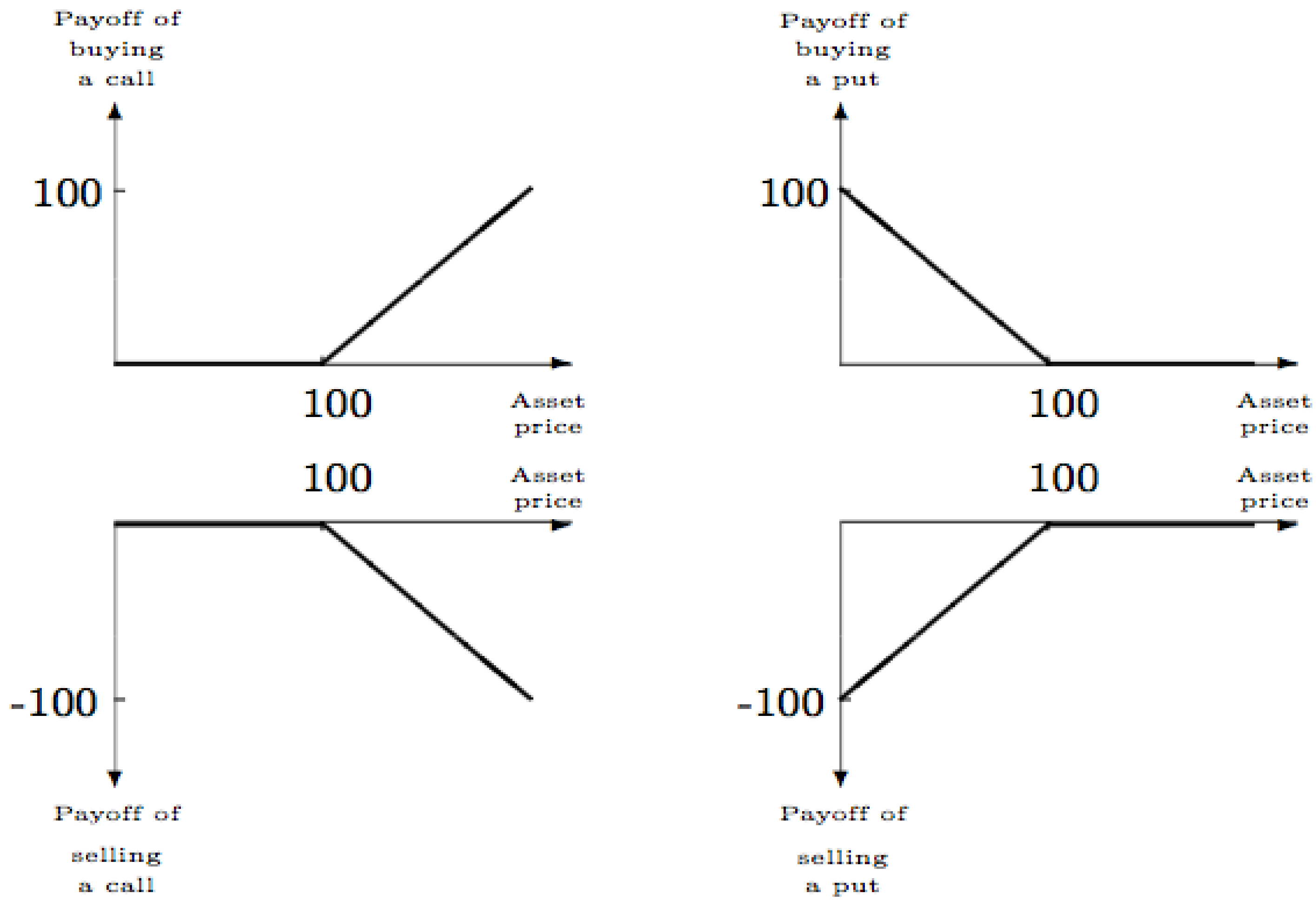
IBM Price at T	Action	Payoff
⋮	Not Exercise	0
80	Not Exercise	0
90	Not Exercise	0
100	Not Exercise	0
110	Exercise	10
120	Exercise	20
130	Exercise	30
⋮	Exercise	$S_T - 100$

_ The payoff of an option is never negative. Sometimes, it is positive.

_ Actual payoff depends on the price of the underlying asset:

$$CF_T \text{ (call)} = \max [S_T - K, 0]$$

Option payoffs can be plotted as a function of the price of the underlying asset at expiration:

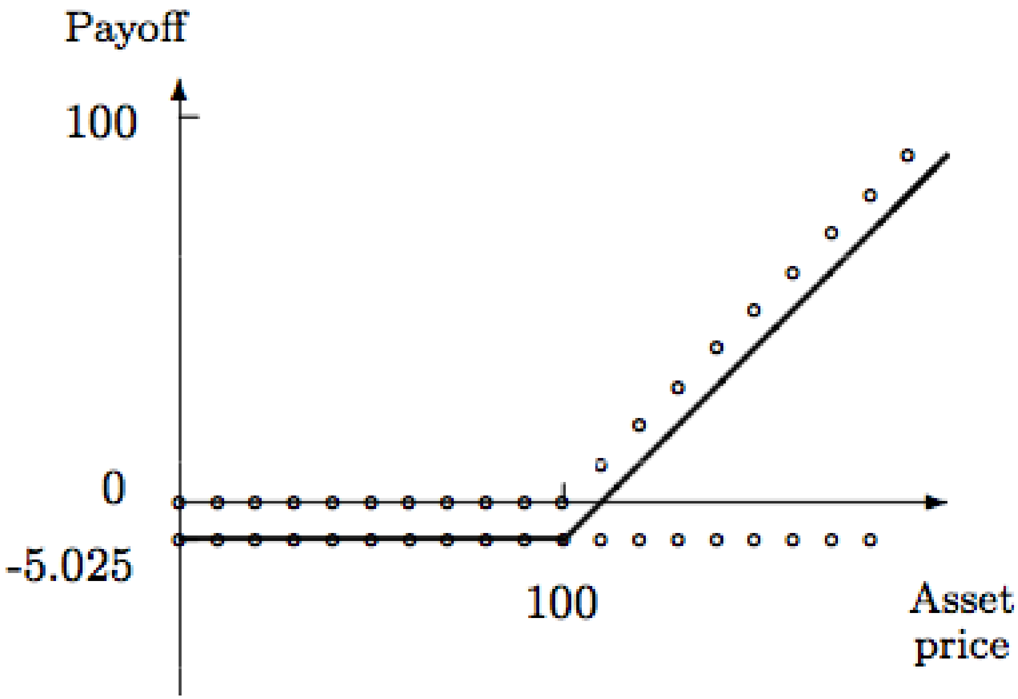


The net payoff from an option must includes its cost.

Example. A European call on IBM shares with an exercise price of \$100 and maturity of three months is trading at \$5. The 3-month interest rate, not annualized, is 0.5%. What is the price of IBM that makes the call break-even?

At maturity, the call's net payoff is as follows:

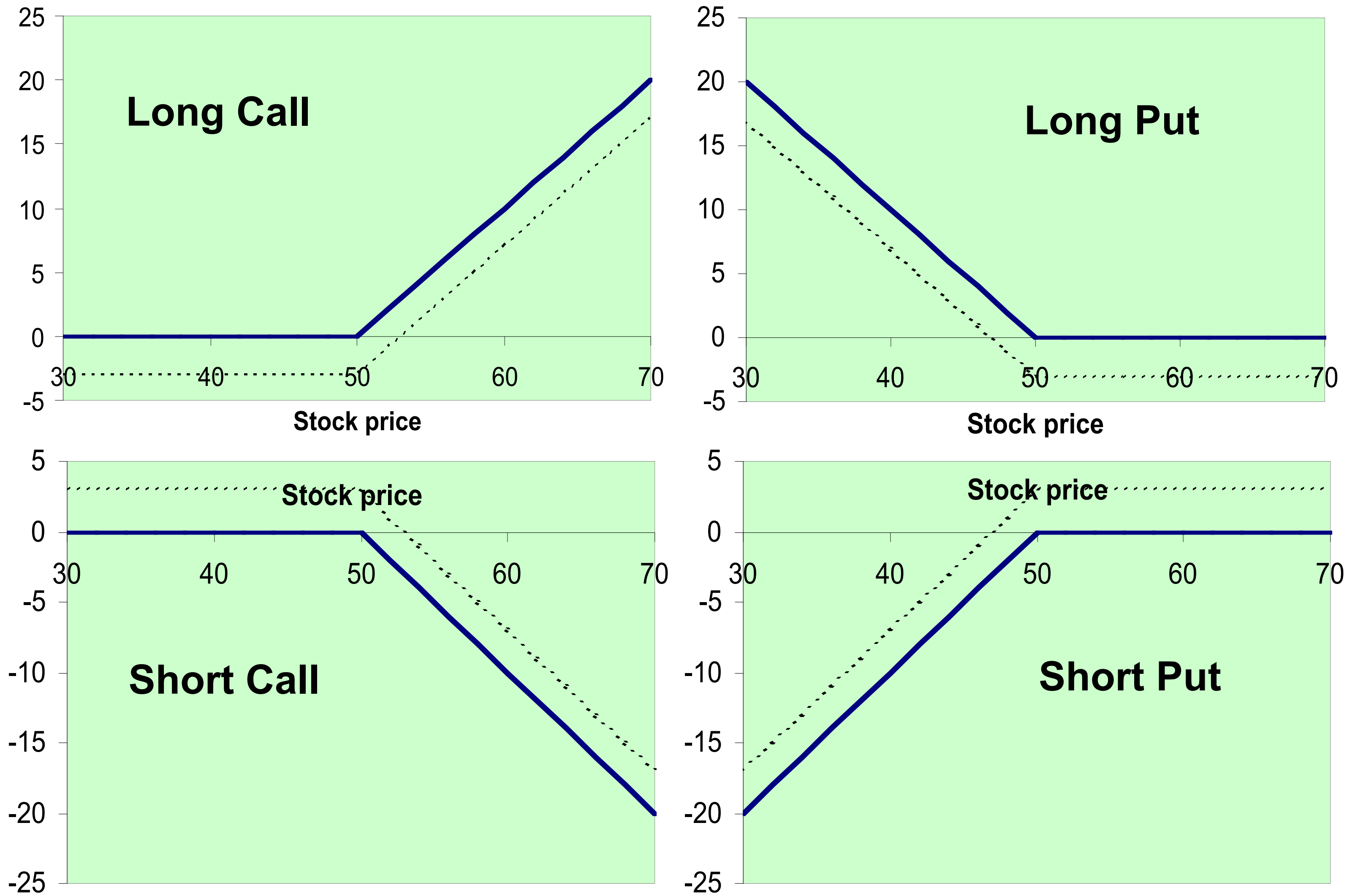
IBM Price	Action	Payoff	Net payoff
⋮	Not Exercise	0	- 5.025
80	Not Exercise	0	- 5.025
90	Not Exercise	0	- 5.025
100	Not Exercise	0	- 5.025
110	Exercise	10	4.975
120	Exercise	20	14.975
130	Exercise	30	24.975
⋮	Exercise	$S_T - 100$	$S_T - 100 - 5.25$



The break even point is given by:

$$\begin{aligned} \text{Net payoff} &= \max[S_T - K,\ 0] - C(1+r)^T \\ &= S_T - 100 - (5)(1+0.005) \\ &= 0 \end{aligned}$$

or $$S_T = \$105.025$$


Long Call
Long Put
Short Call
Short Put
Stock price
Stock price
Stock price
Stock price
25
20
15
10
5
0
-5
-10
-15
-20
-25
30
40
50
60
70

Call option (price = C)

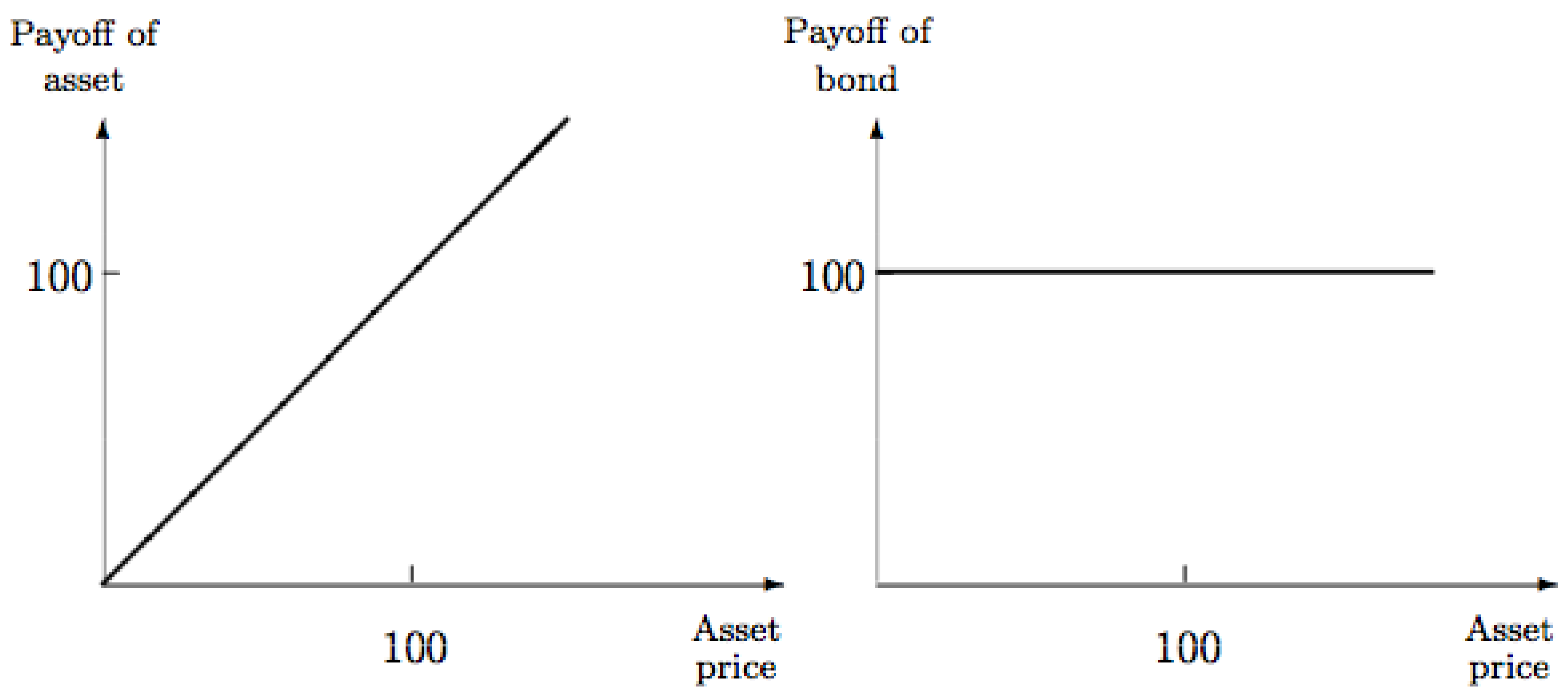
	if $S < K$	if $S = K$	if $S > K$
Payoff	0	0	$S - K$
Profit	$-C(1+r)^T$	$-C(1+r)^T$	$S - K - C(1+r)^T$

Put option (price = P)

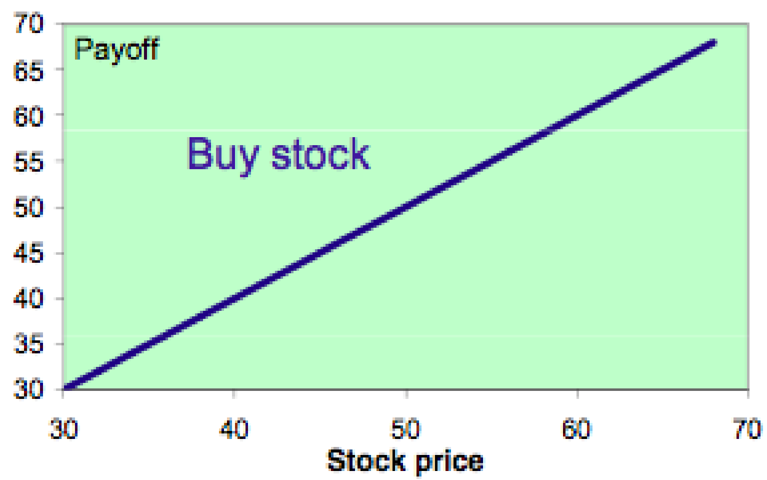
	if $S < K$	if $S = K$	if $S > K$
Payoff	$K - S$	0	0
Profit	$K - S - P(1+r)^T$	$-P(1+r)^T$	$-P(1+r)^T$

Using the payoff diagrams, we can also examine the payoff of a portfolio consisting of options as well as other assets.

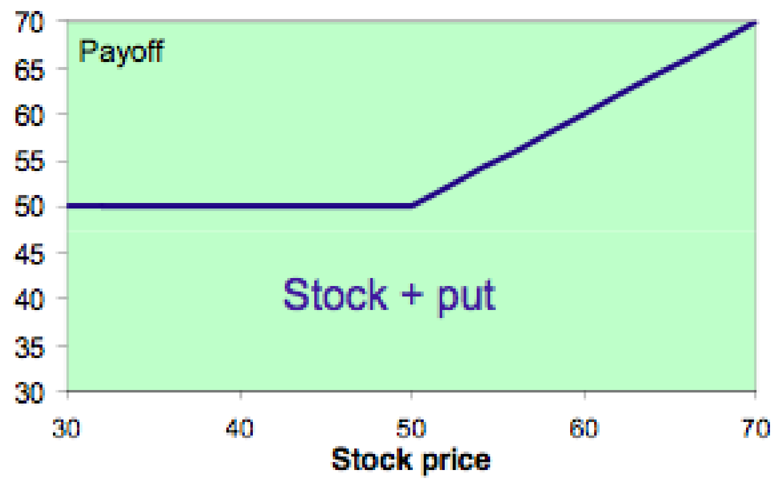
Example. The underlying asset and the bond (with face value $100) have the following payoff diagram:



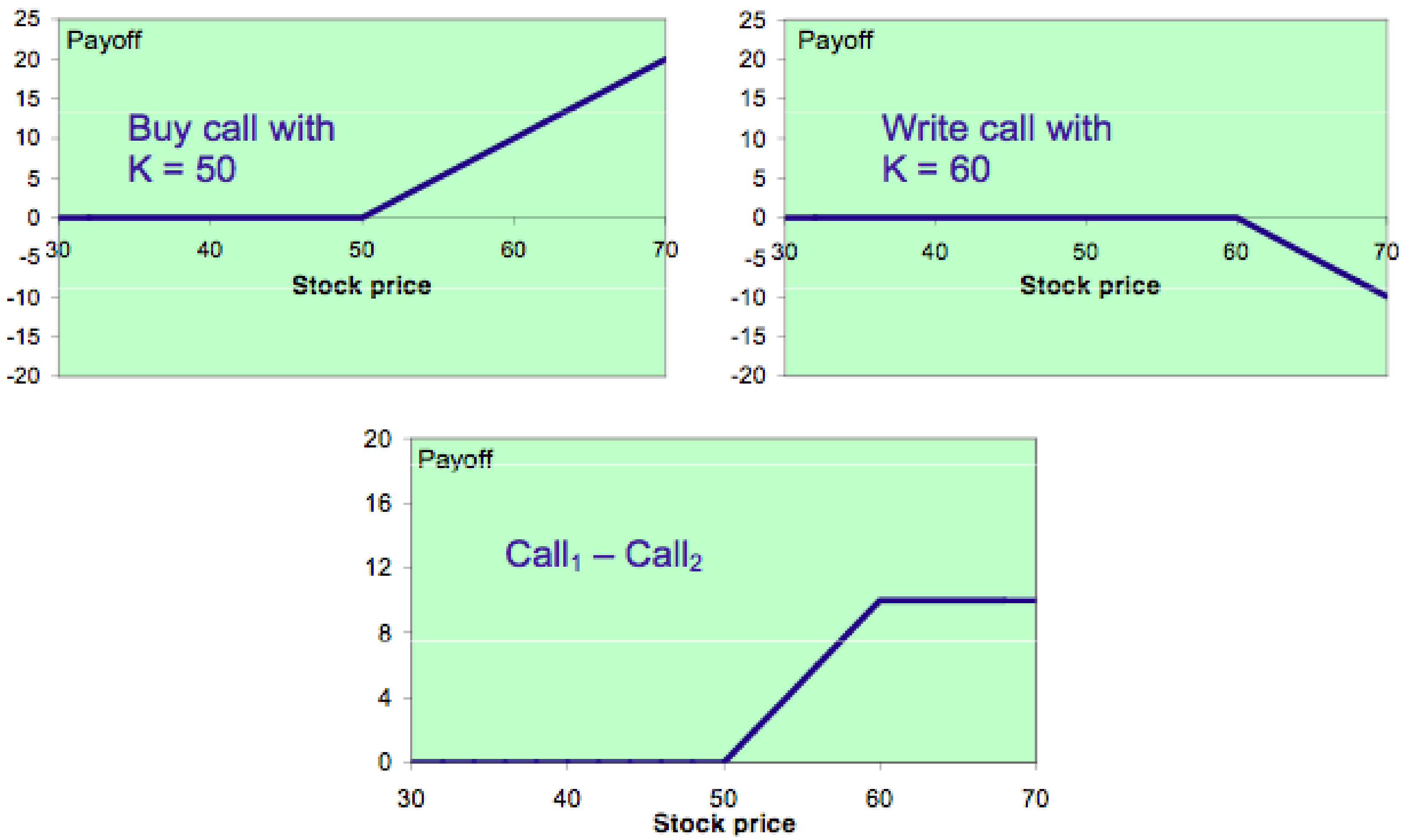
Stock + put


Payoff
Buy stock
Stock price
30
40
50
60
70
35
45
55
65

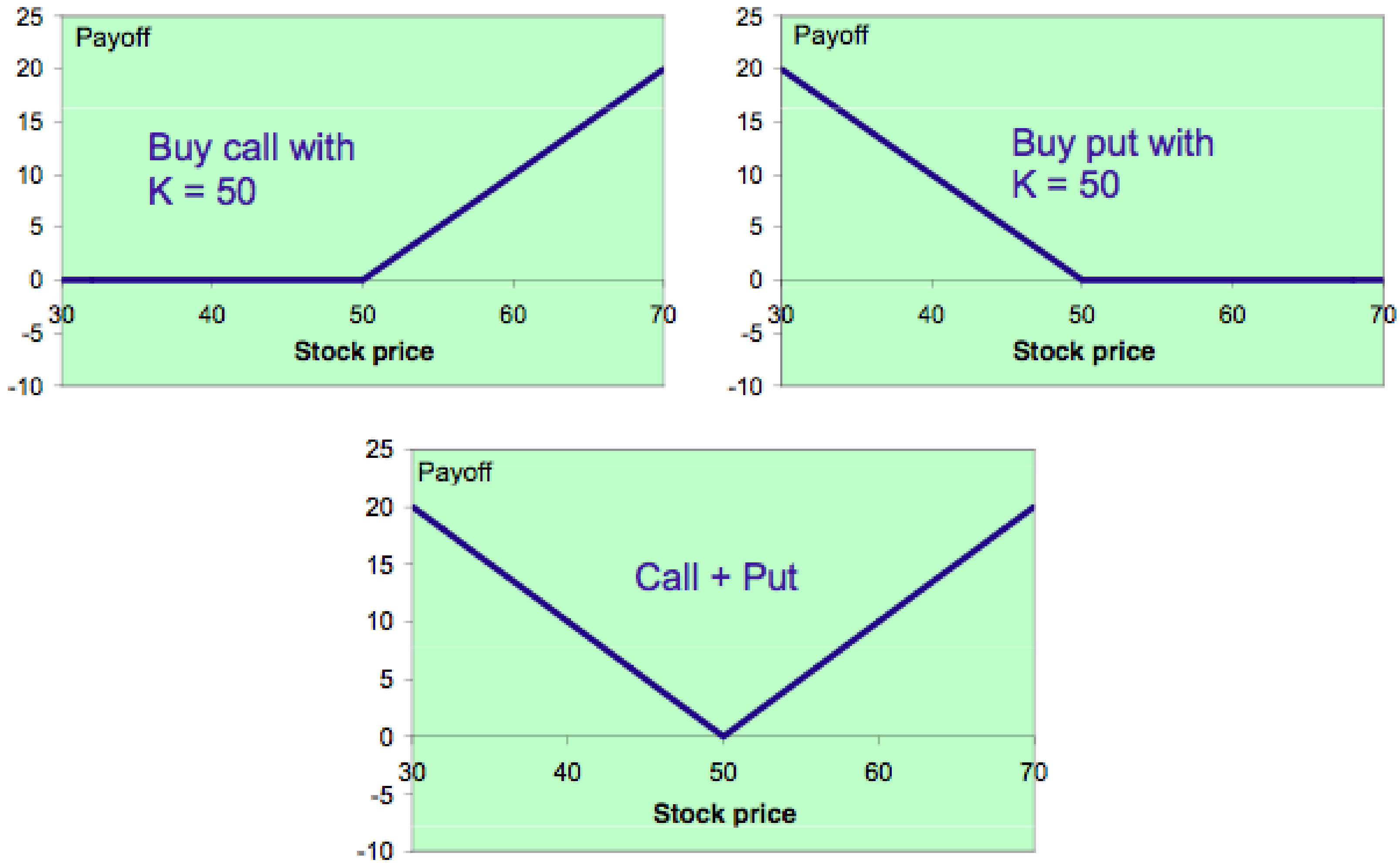

Payoff
Buy a put
Stock price
0
5
10
15
20
25
30
40
50
60
70


Payoff
Stock + put
Stock price
30
40
50
60
70
35
45
55
65

Call 1 - Call 2


Payoff
Buy call with
K = 50
25
20
15
10
5
0
-5
-10
-15
-20
30
40
50
60
70
Stock price
Payoff
Write call with
K = 60
25
20
15
10
5
0
-5
-10
-15
-20
30
40
50
60
70
Stock price
Payoff
$Call_1 – Call_2$
20
16
12
8
4
0
30
40
50
60
70
Stock price

Call + Put


Payoff
Buy call with
K = 50
Stock price
Payoff
Buy put with
K = 50
Stock price
Payoff
Call + Put
Stock price
25
20
15
10
5
0
-5
-10
30
40
50
60
70

Example. Consider two firms, A and B, with identical assets but different capital structures (in market value terms).

Balance sheet of A

Asset	$30	$0	Bond
		30	Equity
	---	---	
	$30	$30	

Balance sheet of B

Asset	$30	$25	Bond
		5	Equity
	---	---	
	$30	$30	

_ Firm B's bond has a face value of $50. Thus default is likely.

Example. (Cont'd)

_ Consider the value of stock A, stock B, and a call on the underlying asset of firm B with an exercise price of $50:

Asset value	Value of stock A	Value of stock B	Value of call
⋮	⋮	⋮	⋮
$20	20	0	0
40	40	0	0
50	50	0	0
60	60	10	10
80	80	30	30
100	100	50	50
⋮	⋮	⋮	⋮

_ Stock B gives the same payoff as a call option written on its asset

_ Thus B's common stocks really are call options of firm's asset

Indeed, many corporate securities can be viewed as options:

Common stock: A call option on the assets of the firm with the exercise price being its bond's redemption value.

Bond: A portfolio combining the firm's assets and a short position in the call with exercise price equal bond redemption value.

$$\text{Equity} \equiv \text{Max}\,[\,0,\ A - B\,]$$

$$\text{Debt} \equiv \text{Min}\,[\,A,\ B\,] = A - \text{Max}\,[\,0,\ A - B\,]$$

$$A = D + E$$

Warrant:	Call options on the stock issued by the firm.
Convertible bond:	A portfolio combining straight bonds and a call option on the firm's stock with the exercise price related to the conversion ratio.
Callable bond:	A portfolio combining straight bonds and a call written on the bonds.

For convenience, we refer to the underlying asset as stock. It could also be a bond, foreign currency or some other asset.

Notation:

- S: Price of stock now
- S_T: Price of stock at T
- B: Price of discount bond of par $1 and maturity T ($B \leq 1$)
- C: Price of a European call with strike K and maturity T
- P: Price of a European put with strike K and maturity T

For our discussions:

_ Consider only European options (no early exercise)

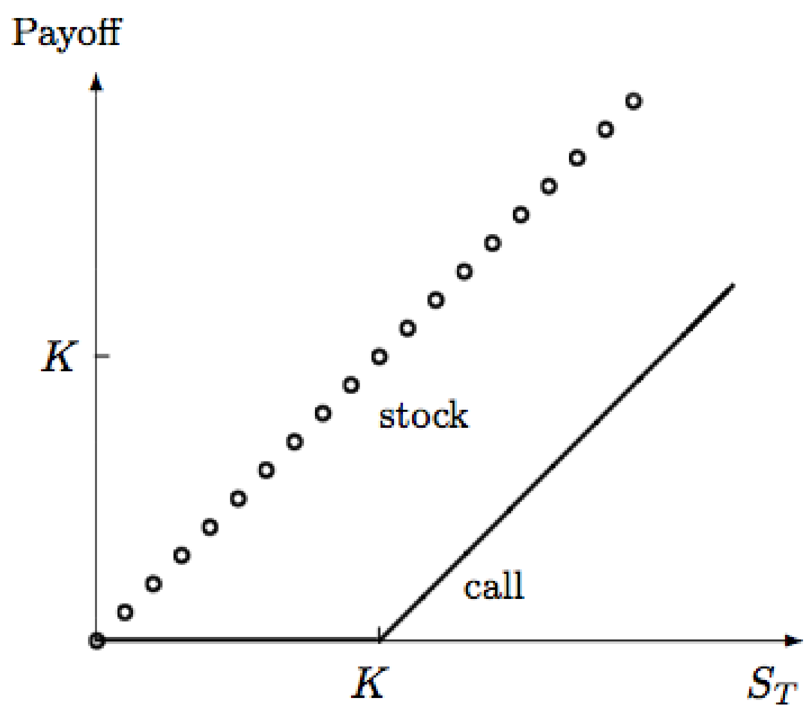
_ Assume no dividends (option cash flow occurs only at maturity)

First consider European options on a non-dividend paying stock.

1. $C \geq 0$

2. $C \leq S$

The payoff of stock dominates that of call:

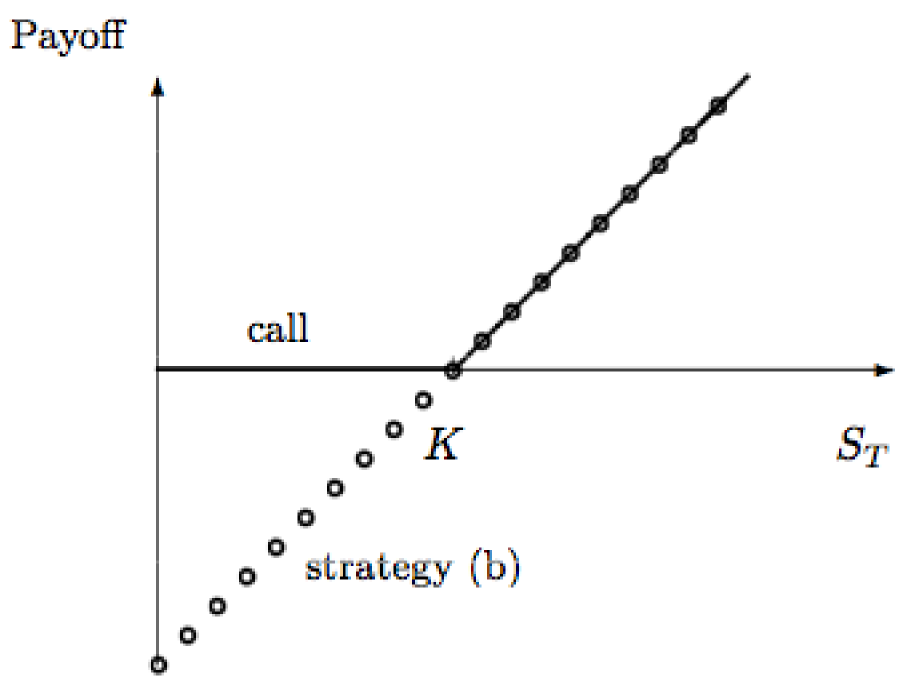


3. $C \geq S - K\,B$

Strategy (a): Buy a call

Strategy (b): Buy a share of stock by borrowing KB

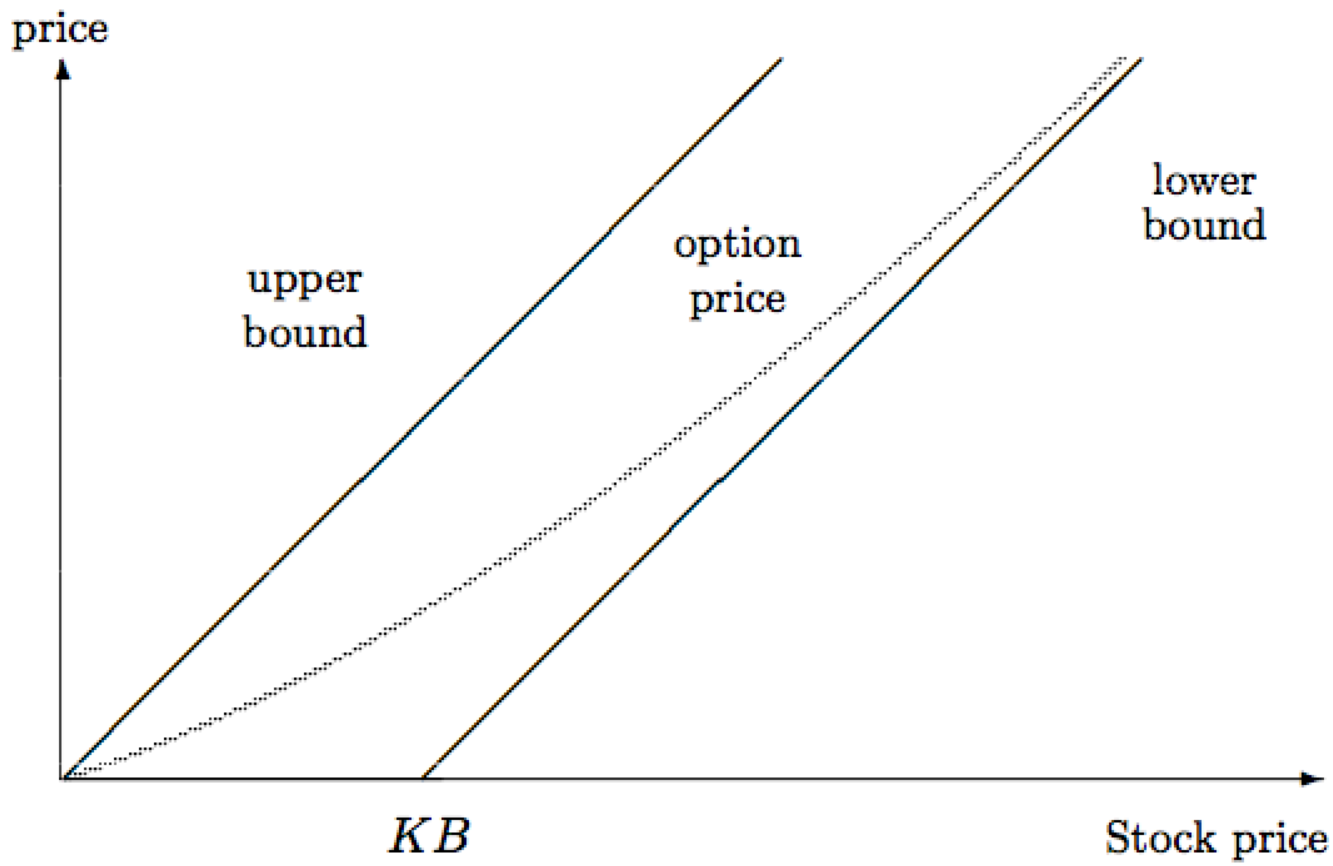
The payoff of strategy (a) dominates that of strategy (b):



Since $C \geq 0$, we have $C \geq \max[S - KB,\ 0]$

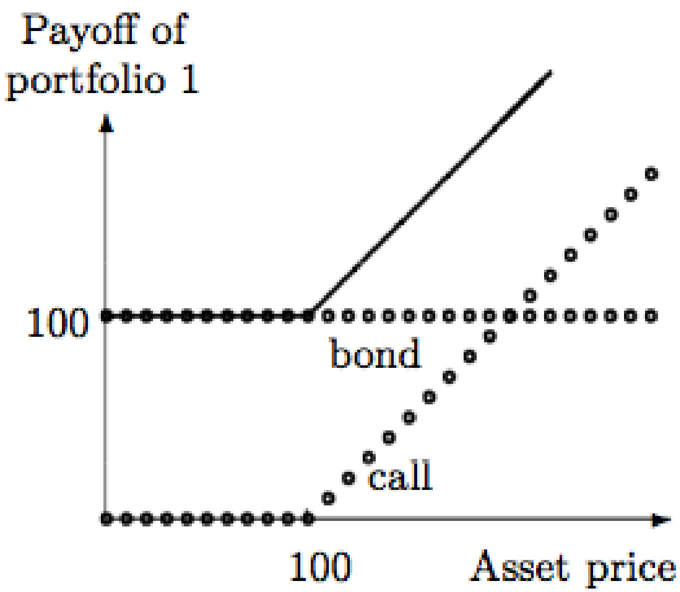
4. Combining the above, we have

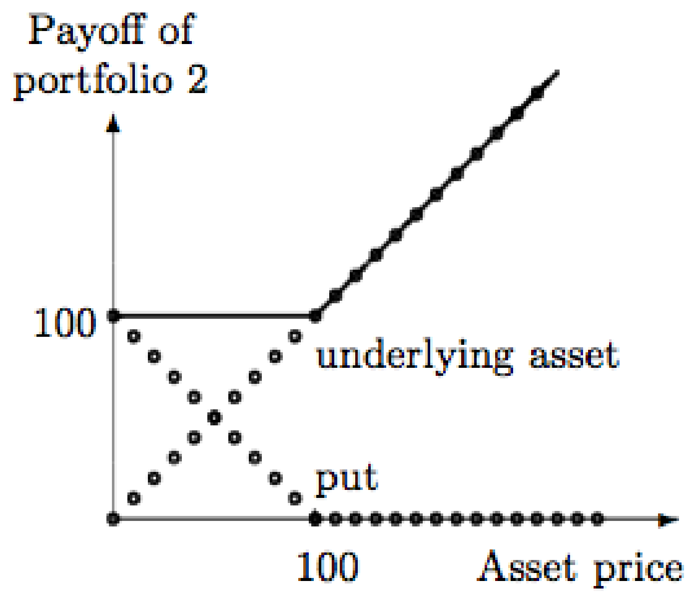
$$\max\,[\,S - KB,\ 0\,] \le C \le S$$



Portfolio 1: A call with strike \$100 and a bond with par \$100

Portfolio 2: A put with strike \$100 and a share of the underlying asset





Their payoffs are identical, so must be their prices:

$$C + K/(1+r)^T = P + S$$

This is called the put-call parity.

Option value increases with the volatility of underlying asset.

Example. Two firms, A and B, with the same current price of $100. B has higher volatility of future prices. Consider call options written on A and B, respectively, with the same exercise price $100.

	Good state	bad state
Probability	p	$1-p$
Stock A	120	80
Stock B	150	50
Call on A	20	0
Call on B	50	0

Clearly, call on stock B should be more valuable.

Determinants of option value:

Key factors in determining option value:

1. price of underlying asset S
2. strike price K
3. time to maturity T
4. interest rate r
5. volatility of underlying asset $\frown$

Additional factors that can sometimes influence option value:

6. expected return on the underlying asset
7. investors' attitude toward risk, …

In order to have a complete option pricing model, we need to make additional assumptions about

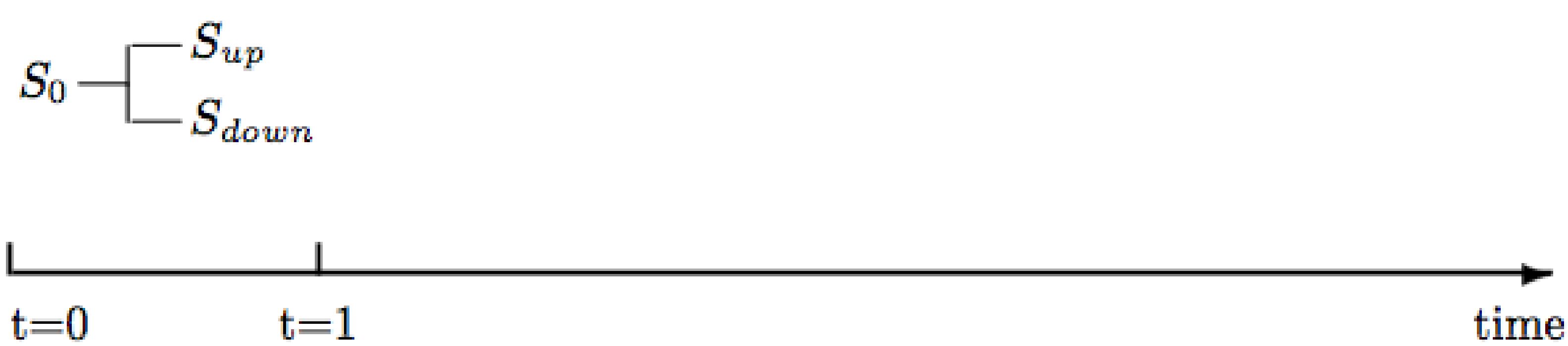
1. Price process of the underlying asset (stock)
2. Other factors

We will assume, in particular, that:

_ Prices do not allow arbitrage

_ Prices are ``reasonable''

_ A benchmark model --- Price follows a binomial process.



Example. Valuation of a European call on a stock.

_ Current stock price is $50

_ There is one period to go

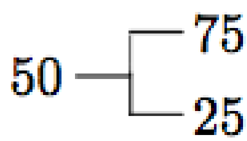
_ Stock price will either go up to $75 or go down to $25

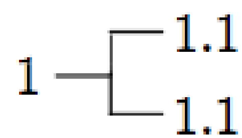
_ There are no cash dividends

_ The strike price is $50

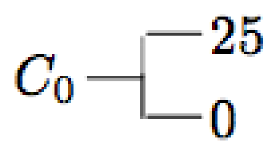
_ One period borrowing and lending rate is 10%

The stock and bond present two investment opportunities:





The option's payoff at expiration is:



What is C_0, the value of the option today?

Form a portfolio of stock and bond that replicates the call's payoff:

- a shares of the stock
- b dollars in the riskless bond

such that

$$\begin{aligned} 75a + 1.1b &= 25 \\ 25a + 1.1b &= 0 \end{aligned}$$

Unique solution: $a = 0.5 \quad \text{and} \quad b = -11.36$

That is

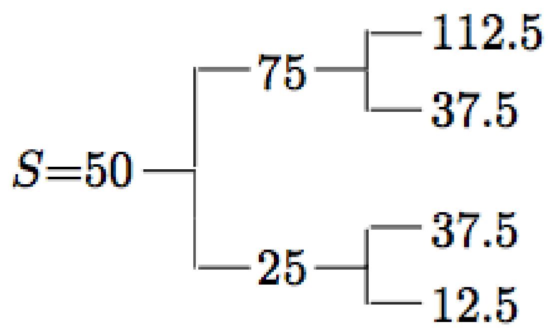
- buy half a share of stock and sell $11.36 worth of bond
- payoff of this portfolio is identical to that of the call
- present value of the call must equal the current cost of this ``replicating portfolio'' which is

$$(50)(0.5) - 11.36 = 13.64$$

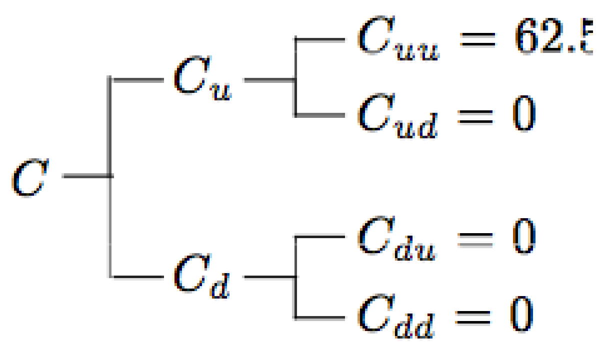
Number of shares needed to replicate one call option is called the option's hedge ratio or delta.

In the above problem, the option's delta is a = 1/2.

More than one period:



Call price process:



- terminal value of the call is known, and
- C_u and C_d denote the option value next period when the stock price goes up and goes down, respectively
- Compute the current value by working backwards: first C_u and C_d and then C

Step 1. Start with Period 1:

1. Suppose the stock price goes up to $75 in period 1:

_ Construct the replicating portfolio at node (t =1, up):

$$\begin{aligned} 112.5a + 1.1b &= 62.5 \\ 37.5a + 1.1b &= 0 \end{aligned}$$

_ Unique solution: a = 0.833, b = - 28.4

_ The cost of this portfolio: (0.833)(75) – 28.4 = 34.075

_ The exercise value of the option: 75 – 50 = 25 ≤ 34.075

_ Thus, C_u = 34.075.

2. Suppose the stock price goes down to $25 in period 1. Repeat the above for node (t =1, down):

_ The replicating portfolio: a = 0, b = 0

_ The call value at the lower node next period is C_d = 0.

Step 2. Now go back one period, to Period 0:

_ The option's value next period is either 34.075 or 0:

$$C_0 \begin{cases} C_u = 34.075 \\ C_d = 0 \end{cases}$$

_ If we can construct a portfolio of the stock and bond to replicate the value of the option next period, then the cost of this replicating portfolio must equal the option's present value.

_ Find a and b so that

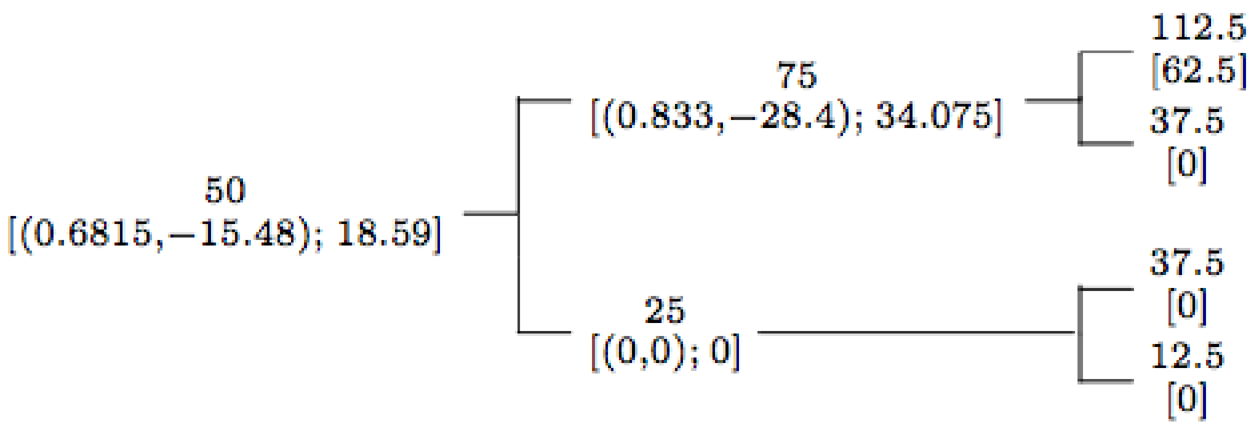
$$\begin{aligned} 75a + 1.1b &= 34.075 \\ 25a + 1.1b &= 0 \end{aligned}$$

_ Unique solution: a = 0.6815, b = -15.48

_ The cost of this portfolio: (0.6815)(50) – 15.48 = 18.59

_ The present value of the option must be C_0 = 18.59

_ It is greater than the exercise value 0 (thus no early exercise)



Play Forward:

Period 0: Spend \$18.59 and borrow \$15.48 at 10% interest rate to buy 0.6815 shares of the stock

Period 1:

- When the stock price goes up, the portfolio value becomes 34.075. Re-balance the portfolio to include 0.833 stock shares, financed by borrowing 28.4 at 10%
  - ✓ One period later, the payoff of this portfolio exactly matches that of the call
- When the stock price goes down, the portfolio becomes worthless. Close out the position.
  - ✓ The portfolio payoff one period later is zero

Thus

_ No early exercise.

_ Replicating strategy gives payoffs identical to those of the call.

_ Initial cost of the replicating strategy must equal the call price.

What we have used to calculate option's value:

- current stock price
- magnitude of possible future changes of stock price -- volatility
- interest rate
- strike price
- time to maturity

What we have not used:

- probabilities of upward and downward movements
- investor's attitude towards risk

Questions on the Binomial Model

- What is the length of a period?
- Price can take more than two possible values.
- Trading takes place continuously.

If we let the period-length get smaller and smaller, we obtain the Black-Scholes option pricing formula:

$$C(S,K,T) = SN\left(x\right) - KR^{-T}N\left(x - \sigma\sqrt{T}\right)$$

where

_ x is defined by

$$x = \frac{\ln\left(S/KR^{-T}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

_ T is in units of a year

_ R = 1+r, where r is the annual riskless interest rate

_ σ is the volatility of annual returns on the underlying asset

_ N (.) is the cumulative normal density function

An interpretation of the Black-Scholes formula:

_ The call is equivalent to a levered long position in the stock.

_ *S N(x)* is the amount invested in the stock

_ $KR^{-T}N\left(x - \sigma\sqrt{T}\right)$ is the dollar amount borrowed

_ The option delta is $N(x) = C_S$

Example. Consider a European call option on a stock with the following data:

_ S = 50, K = 50, T = 30 days

_ The volatility σ is 30% per year

_ The current annual interest rate is 5.895%

Then

$$x = \frac{\ln\left(50/50(1.05895)^{-\frac{30}{365}}\right)}{(0.3)\sqrt{\frac{30}{365}}} + \frac{1}{2}(0.3)\sqrt{\frac{30}{365}} = 0.0977$$

$$\begin{aligned} C &= 50N(0.0977) - 50(1.05895)^{-\frac{30}{365}} N\left(0.0977 - 0.3\sqrt{\frac{30}{365}}\right) \\ &= 50(0.53890) - 50(0.99530)(0.50468) \\ &= 1.83 \end{aligned}$$

- Introduction to options
- Option payoffs
- Corporate securities as options
- Use of options
- Basic properties of options
- Binomial Option Pricing Model
- Black-Scholes option pricing formula