15.401 Finance Theory I

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Lecture 8: Portfolio Theory

- Portfolios
- Portfolio returns
- Diversification
- Systematic vs. non-systematic risks
- Optimal portfolio choices
- Sharpe ratio

Readings:

- Brealey, Myers and Allen, Chapter 8, 9.1
- Bodie, Kane and Markus, Chapters 6.2, 7, 8

What is a portfolio?

A portfolio is simply a collections of assets:

- n assets, each with share price P_i (i = 1, 2, … n)
- A portfolio is a collection of N_i shares of each asset i
- Total value of portfolio:

$$V = N_1 P_1 + N_2 P_2 + \cdots + N_n P_n = \sum_{i=1}^{n} N_i P_i$$

- A typical portfolio has V > 0. Define portfolio weights:

$$w_i = \frac{N_i P_i}{N_1 P_1 + N_2 P_2 + \cdots + N_n P_n} = \frac{N_i P_i}{V}$$

- A portfolio can then also be defined by its asset weights

$$\{w_1, w_2, \ldots, w_n\}, \quad w_1 + w_2 + \cdots + w_n = 1$$

Example. Your investment account of $100,000 consists of three stocks: 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	10%
B	1,000	$60	$60,000	60%
C	750	$40	$30,000	30%
Total			$100,000	100%

Example (cont). Your broker informs you that you only need to keep \$50,000 in your investment account to support the same portfolio of 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C; in other words, you can buy these stocks on margin. You withdraw \$50,000 to use for other purposes, leaving \$50,000 in the account. Your portfolio is summarized by the following weights:

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	\$50	\$10,000	20%
B	1,000	\$60	\$60,000	120%
C	750	\$40	\$30,000	60%
Riskless Bond	–\$50,000	\$1	–\$50,000	–100%
Total			\$50,000	100%

Example. You decide to purchase a home that costs $500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80% What are your portfolio weights for this investment?

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
Home	1	$500,000	$500,000	500%
Mortgage	1	–$400,000	–$400,000	–400%
Total			$100,000	100%

Leverage ratio = asset/net investment = 5

What happens to your total assets if your home price declines by 15%?

Why not pick the best asset instead of forming a portfolio?

- We don't know which stock is best!
- Portfolios provide diversification, reducing unnecessary risks
- Portfolios can enhance performance by focusing bets
- Portfolios can customize and manage risk/reward trade-offs

How do we chose a "good" portfolio?

- What does "good" mean?
- What characteristics of a portfolio do we care about?
  - risk and reward (expected return)
  - higher expected returns are preferred
  - higher risks are not preferred

A portfolio's characteristics are determined by the returns of its assets and its weights in them.

– Mean returns:

Asset	1	2	$\cdots$	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	$\cdots$	$\bar{r}_n$

– Variances and co-variances:

	r_1	r_2	$\cdots$	r_n
r_1	σ_1^2	σ_{12}	$\cdots$	σ_{1n}
r_2	σ_{21}	σ_2^2	$\cdots$	σ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
r_n	σ_{n1}	σ_{n2}	$\cdots$	σ_n^2

Covariance of an asset with itself is its variance:

Example. Monthly stock returns on IBM (r_1) and Merck (r_2):

Mean returns

$\bar{r}_1$	$\bar{r}_2$
0.0149	0.0100

Covariance matrix

	$\tilde{r}_1$	$\tilde{r}_2$
$\tilde{r}_1$	0.007770	0.002095
$\tilde{r}_2$	0.002095	0.003587

Note: σ_1 = 8.81%, σ_2 = 5.99 and ρ_{12} = 0.40.

The portfolio return is a weighted average of the individual returns:

$$\tilde{r}_p = w_1\tilde{r}_1 + w_2\tilde{r}_2$$

Example. Suppose you invest \$600 in IBM and \$400 in Merck for a month. If the realized return is 2.5% on IBM and 1.5% on Merck over the month, what is the return on your total portfolio?

The portfolio weights are

$$w_{\text{IBM}} = 600/1000 = 60\% \qquad w_{\text{Merck}} = 400/1000 = 40\%$$

$$\begin{aligned} r_p &= \frac{(600)(0.025) + (400)(0.015)}{1000} \\ &= (0.6)(0.025) + (0.4)(0.015) \\ &= 2.1\% \end{aligned}$$

Expected return on a portfolio with two assets

Expected portfolio return:

$$\bar{r}_p = w_1 \bar{r}_1 + w_2 \bar{r}_2$$

Unexpected portfolio return:

$$\tilde{r}_p - \bar{r}_p = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2)$$

Variance of return on a portfolio with two assets

The variance of the portfolio return:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

which is also the sum of all entries of the following table

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$w_1^2\sigma_1^2$	$w_1w_2\sigma_{12}$
$w_2\tilde{r}_2$	$w_1w_2\sigma_{12}$	$w_2^2\sigma_2^2$

Example. Consider again investing in IBM and Merck stocks.

Mean returns

$\bar{r}_1$	$\bar{r}_2$
0.0149	0.0100

Covariance matrix

	$\tilde{r}_1$	$\tilde{r}_2$
$\tilde{r}_1$	0.007770	0.002095
$\tilde{r}_2$	0.002095	0.003587

Consider the equally weighted portfolio: $w_1 = w_2 = 0.5$

Mean of portfolio return: $\bar{r}_p = (0.5)(0.0149) + (0.5)(0.0100) = 1.25\%$

Variance of portfolio return:

$$\begin{aligned} \sigma_p^2 &= (0.5)^2(0.007770) + (0.5)^2(0.003587) + (2)(0.5)^2(0.002095) \\ &= 0.003888 \\ \sigma_p &= 6.23\% \end{aligned}$$

	$w_1\tilde{r}_1$	$w_2\tilde{r}_2$
$w_1\tilde{r}_1$	$(0.5)^2(0.007770)$	$(0.5)^2(0.002095)$
$w_2\tilde{r}_2$	$(0.5)^2(0.002095)$	$(0.5)^2(0.003587)$

– Expected portfolio return:

$$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2 + w_3\bar{r}_3$$

– The variance of the portfolio return:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23}$$

Example. IBM, Merck and Intel returns have covariance matrix:

	$\tilde{r}_{\text{IBM}}$	$\tilde{r}_{\text{Merck}}$	$\tilde{r}_{\text{Intel}}$
$\tilde{r}_{\text{IBM}}$	0.007770	0.002095	0.001189
$\tilde{r}_{\text{Merck}}$	0.002095	0.003587	0.000229
$\tilde{r}_{\text{Intel}}$	0.001189	0.000229	0.009790

– What is the risk (StD) of the equally weighted portfolio?

$$\sigma_p^2 = \left(\tfrac{1}{3}\right)^2 \times (\text{Sum of all entries of covariance matrix}) = 0.003130$$

$$\sigma_p = 5.59\%$$

– For individual assets: $\sigma_{\text{IBM}} = 8.81\%$, $\sigma_{\text{Merck}} = 5.99\%$, $\sigma_{\text{Intel}} = 9.89\%$

We now consider a portfolio of n assets:

$$\{w_1, w_2, \ldots, w_n\}, \quad \sum_i w_i = 1$$

1. The return on the portfolio is:

$$\tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + \cdots + w_n \tilde{r}_n$$

2. The expected return on the portfolio is:

$$\bar{r}_p = \mathrm{E}[r_p] = w_1 \bar{r}_1 + w_2 \bar{r}_2 + \cdots + w_n \bar{r}_n$$

3. The variance of portfolio return is:

$$\sigma_p^2 = \mathrm{Var}[\tilde{r}_p] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}, \quad \sigma_{ii} = \sigma_i^2$$

4. The volatility (StD) of portfolio return is:

$$\sigma_p = \sqrt{\mathrm{V}[\tilde{r}_p]} = \sqrt{\sigma_p^2}$$

The variance of portfolio return can be computed by summing up all the entries to the following table:

	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$w_1 w_2 \sigma_{12}$	$\cdots$	$w_1 w_n \sigma_{1n}$
$w_2 r_2$	$w_2 w_1 \sigma_{21}$	$w_2^2 \sigma_2^2$	$\cdots$	$w_2 w_n \sigma_{2n}$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$w_n w_2 \sigma_{n2}$	$\cdots$	$w_n^2 \sigma_n^2$

The variance of a sum is not just the sum of variances! We also need to account for the covariances.

In order to calculate return variance of a portfolio, we need

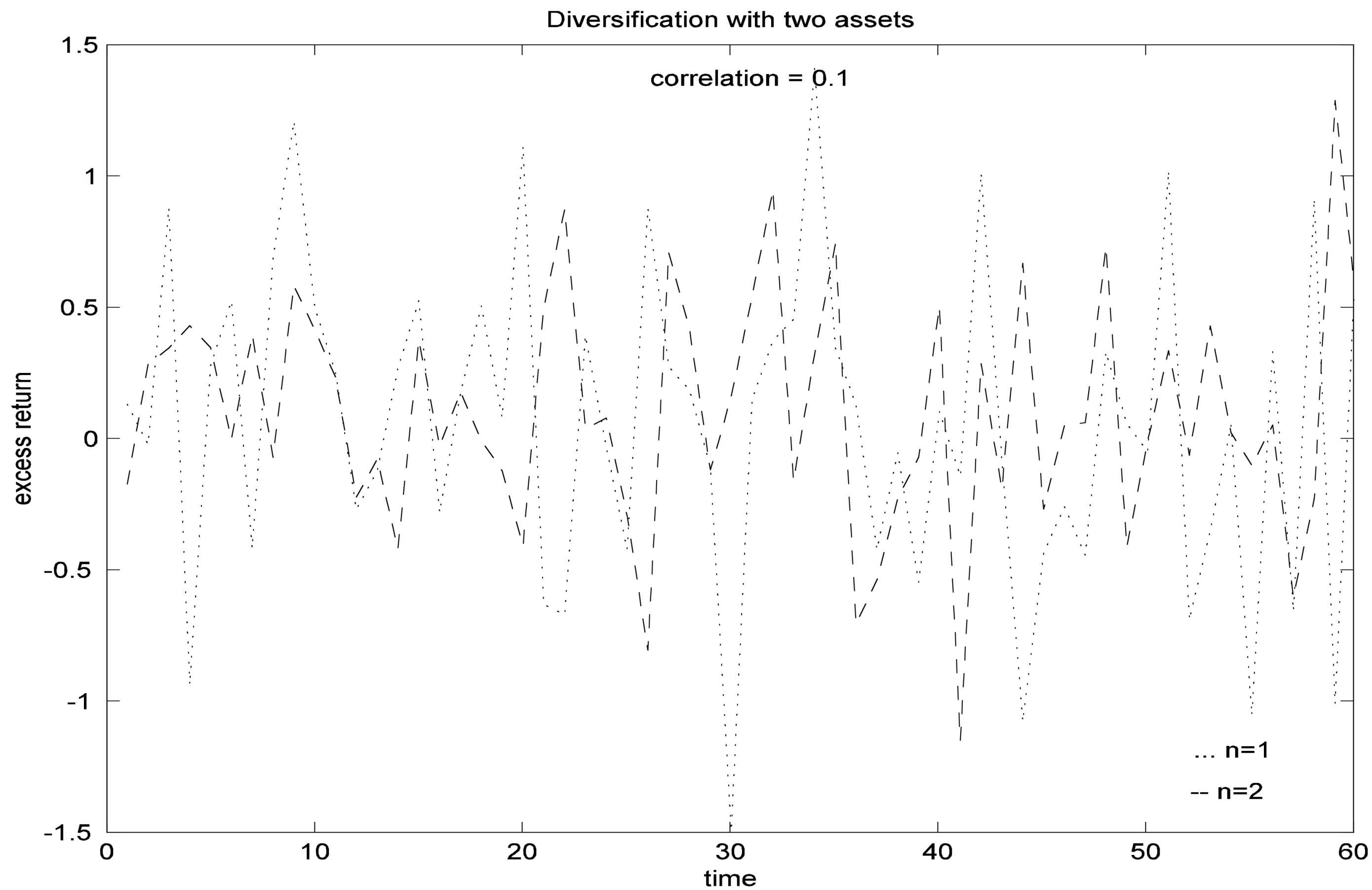
a) portfolio weights

b) individual variances

c) all covariances

Diversification reduces risk.

1. Two assets:

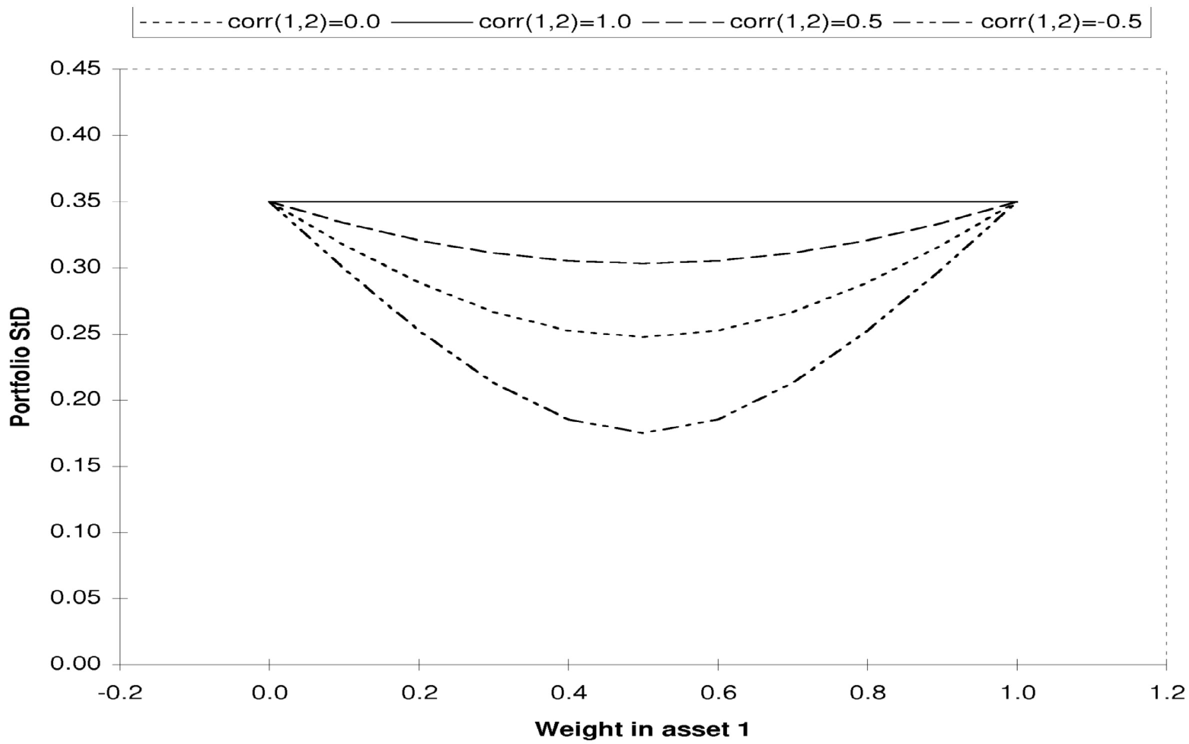


Example. Two assets with the same annual return StD of 35%.

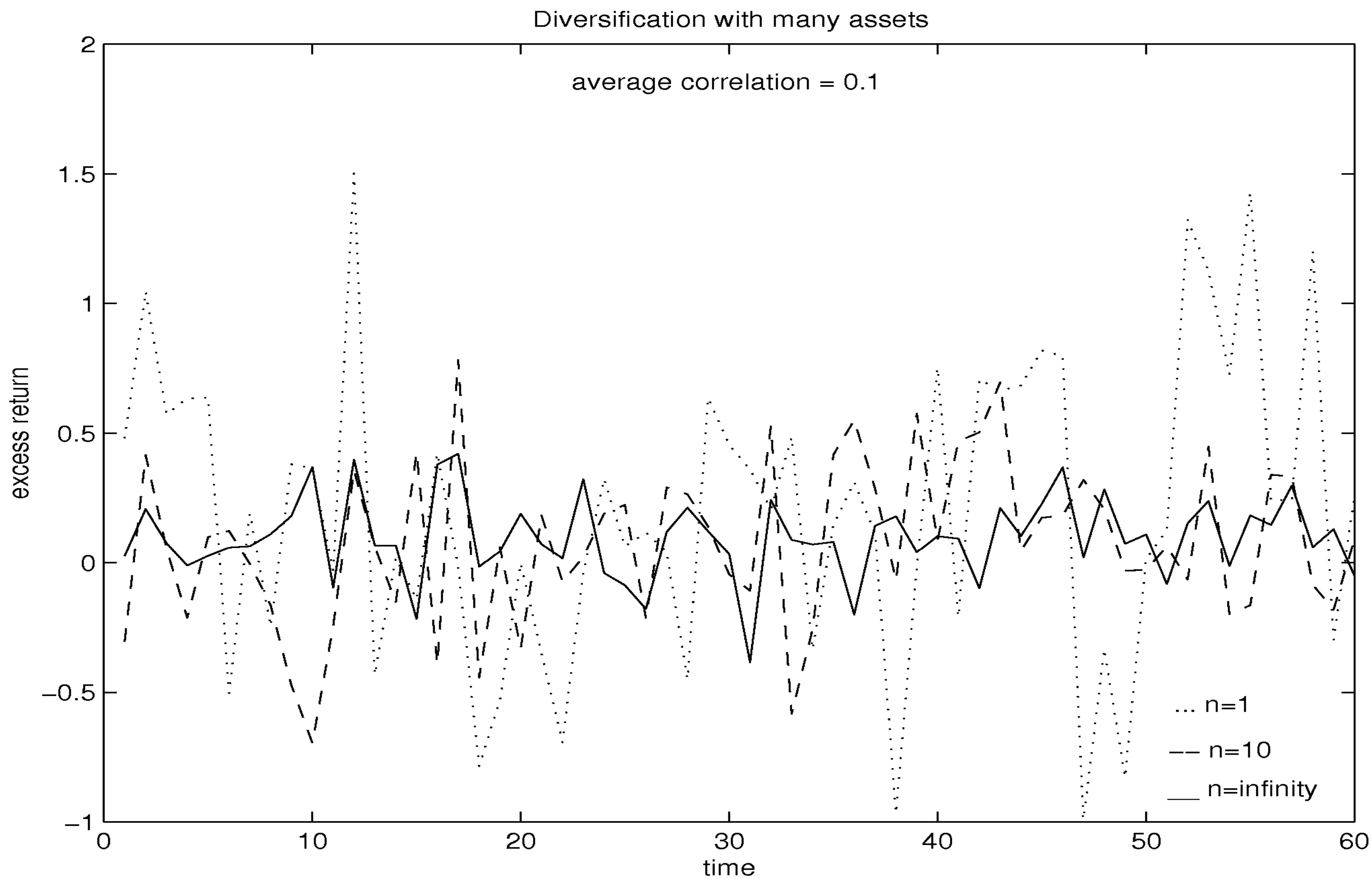
Consider a portfolio p with weight w in asset 1 and 1- w in asset 2.

$$\sigma_p = \sqrt{w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}}.$$

StD of portfolio return is less than the StD of each individual asset.

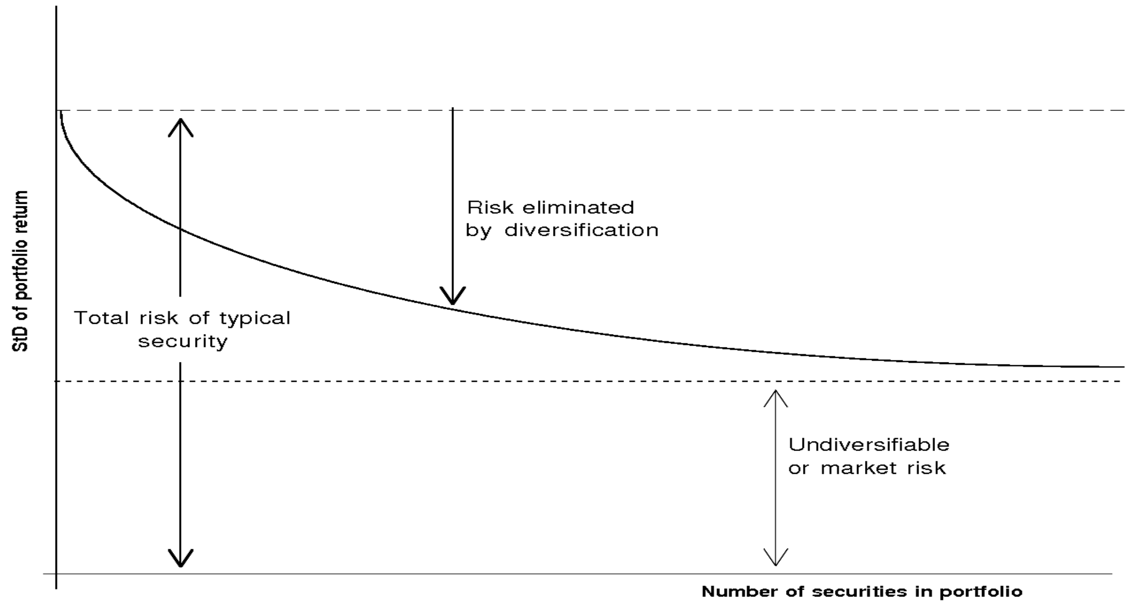


2. Multiple assets:



Certain risks cannot be diversified away.

Impact of diversification on portfolio risk



Risk comes in two types:

- Diversifiable risk
- Non-diversifiable (market, systematic) risk due to macro (business cycle, inflation, etc.) / market conditions (liquidity)

Example. An equally-weighted portfolio of n assets:

	$w_1 r_1$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$\cdots$	$w_1 w_n \sigma_{1n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_n w_1 \sigma_{n1}$	$\cdots$	$w_n^2 \sigma_n^2$

_ A typical variance term: $\left(\frac{1}{n}\right)^2 \sigma_{ii}$.

- There are n variance terms.

_ A typical covariance term: $\left(\frac{1}{n}\right)^2 \sigma_{ij}, (i \neq j)$

- There are n^2-n covariance terms.

Add all the terms:

$$
\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ii} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ij} \\
&= \left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \left(\frac{n^2-n}{n^2}\right)\left(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}\right) \\
&= \left(\frac{1}{n}\right)(\text{average variance}) + \left(\frac{n^2-n}{n^2}\right)(\text{average covariance})
\end{aligned}
$$

As n becomes very large:

_ Contribution of variance terms goes to zero.

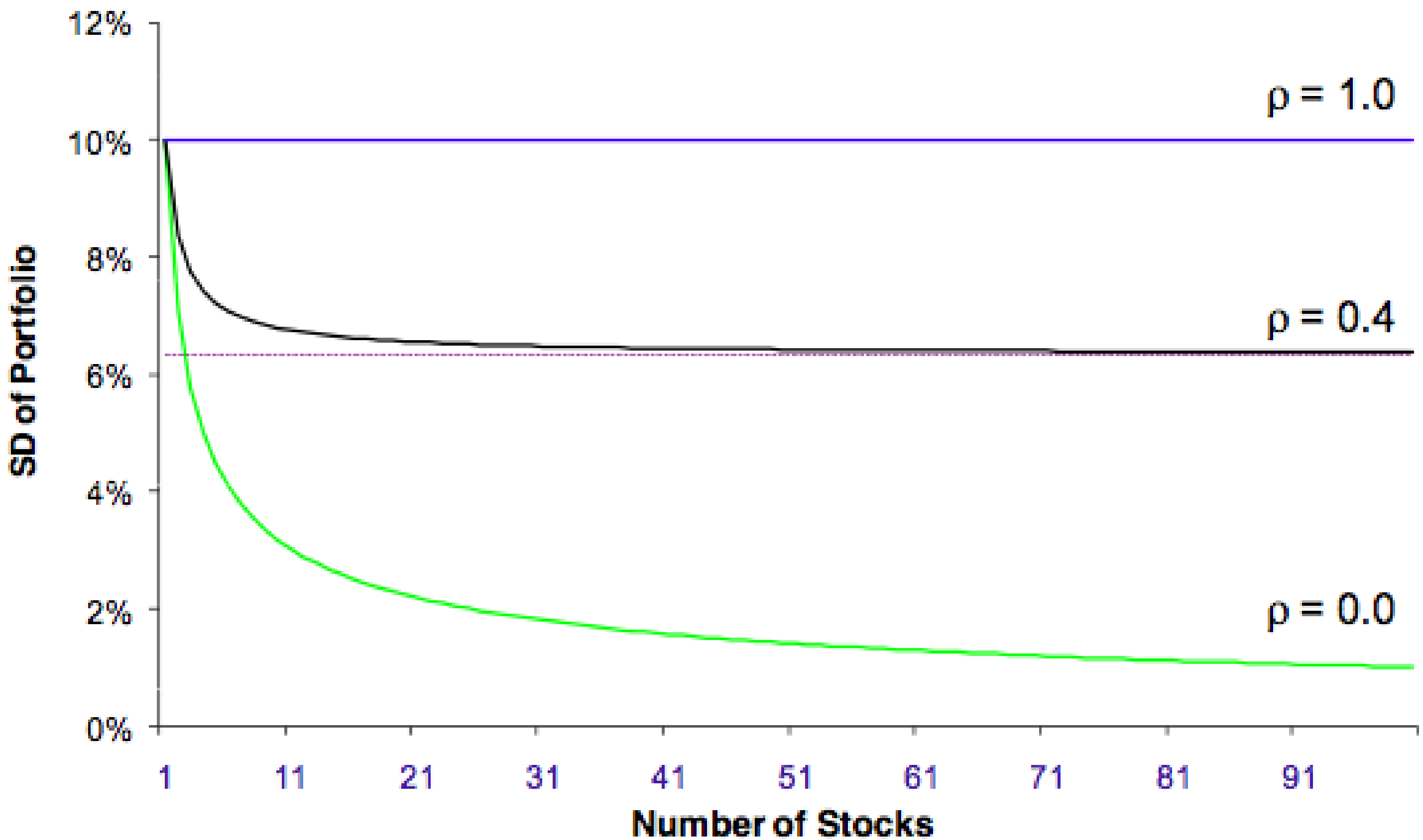
_ Contribution of covariance terms goes to “average covariance”.

Example (ctd). The average stock has a monthly standard deviation of 10% and the average correlation between stocks is 0.4. If you invest the same amount in each stock, what is the variance of the portfolio?

$$
\begin{aligned}
\mathrm{Cov}[R_i, R_j] &= \rho_{ij}\sigma_i\sigma_j = 0.40 \times 0.10 \times 0.10 = 0.004 \\
\mathrm{Var}[R_p] &= \frac{1}{n}0.10^2 + \frac{n-1}{n}0.004 \approx 0.004 \quad \text{if } n \text{ large} \\
\sigma_p &\approx \sqrt{0.004} = 6.3\%
\end{aligned}
$$

What is the correlation is 0? What if it is 1?

Example (ctd).

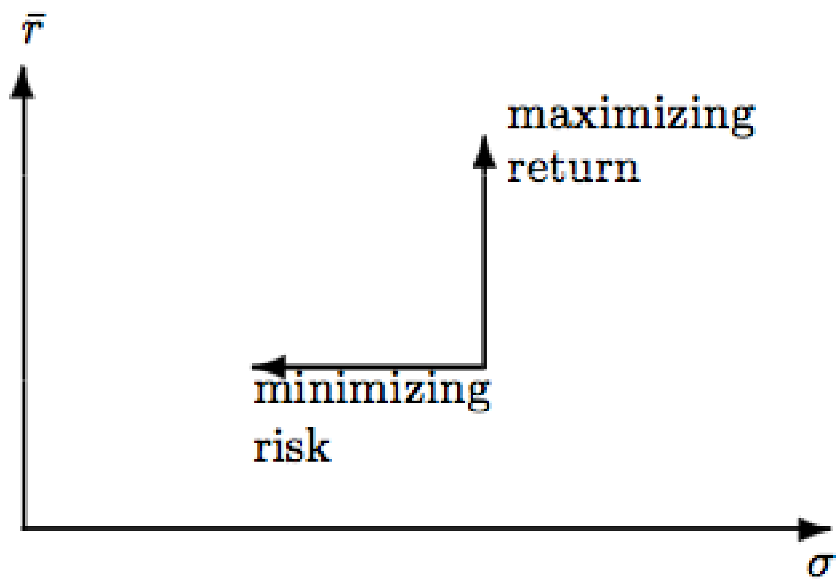


12%
10%
8%
6%
4%
2%
0%
SD of Portfolio
ρ = 1.0
ρ = 0.4
ρ = 0.0
1
11
21
31
41
51
61
71
81
91
Number of Stocks

How to choose a portfolio:

_ Minimize risk for a given expected return? or

_ Maximize expected return for a given risk?



Formally, we need to solve the following problem:

$$
\begin{array}{lll}
(\mathrm{P}): & \underset{\{w_1,\ldots,w_n\}}{\text{Minimize}} & \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
& \text{subject to} & (1) \quad \sum_{i=1}^{n} w_i = 1 \\
& & (2) \quad \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}_p
\end{array}
$$

$$\bar{r}_p = w\bar{r}_1 + (1-w)\bar{r}_2$$

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}$$

Without short sales

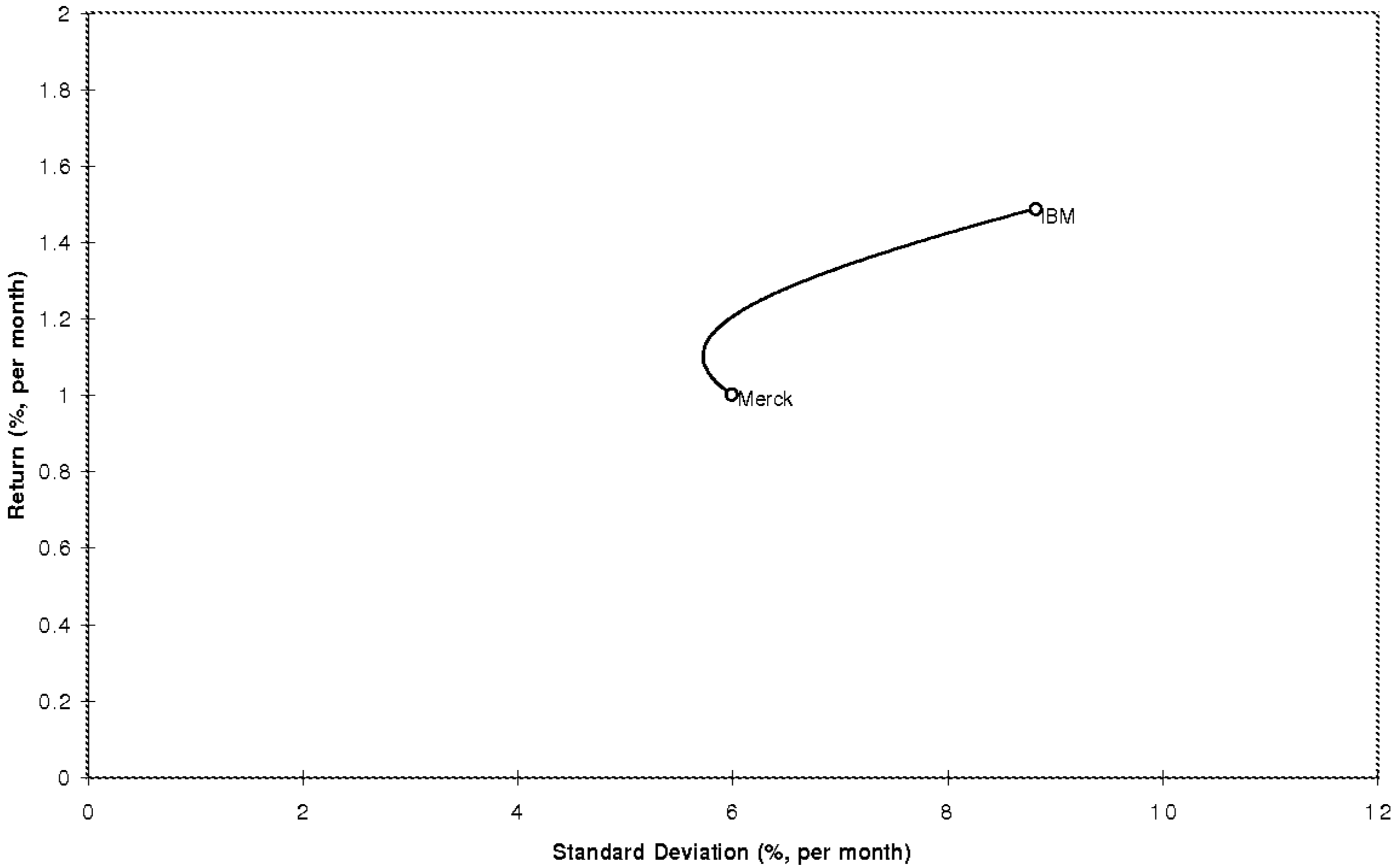
Example (ctd). IBM and Merck:

Covariances	$\tilde{r}_{\mathrm{IBM}}$	$\tilde{r}_{\mathrm{Merck}}$
$\tilde{r}_{\mathrm{IBM}}$	0.007770	0.002095
$\tilde{r}_{\mathrm{Merck}}$	0.002095	0.003587
Mean (%)	1.49	1.00
SD (%)	8.81	5.99

Portfolios of IBM and Merck

Weight in IBM (%)	0	10	20	30	40	50	60	70	80	90	100
Mean return (%)	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.34	1.39	1.44	1.49
SD (%)	5.99	5.80	5.72	5.78	5.95	6.23	6.62	7.08	7.61	8.19	8.81

Portfolio frontier when short sales re not allowed



With short sales

When short sales are allowed, portfolio weights are unrestricted.

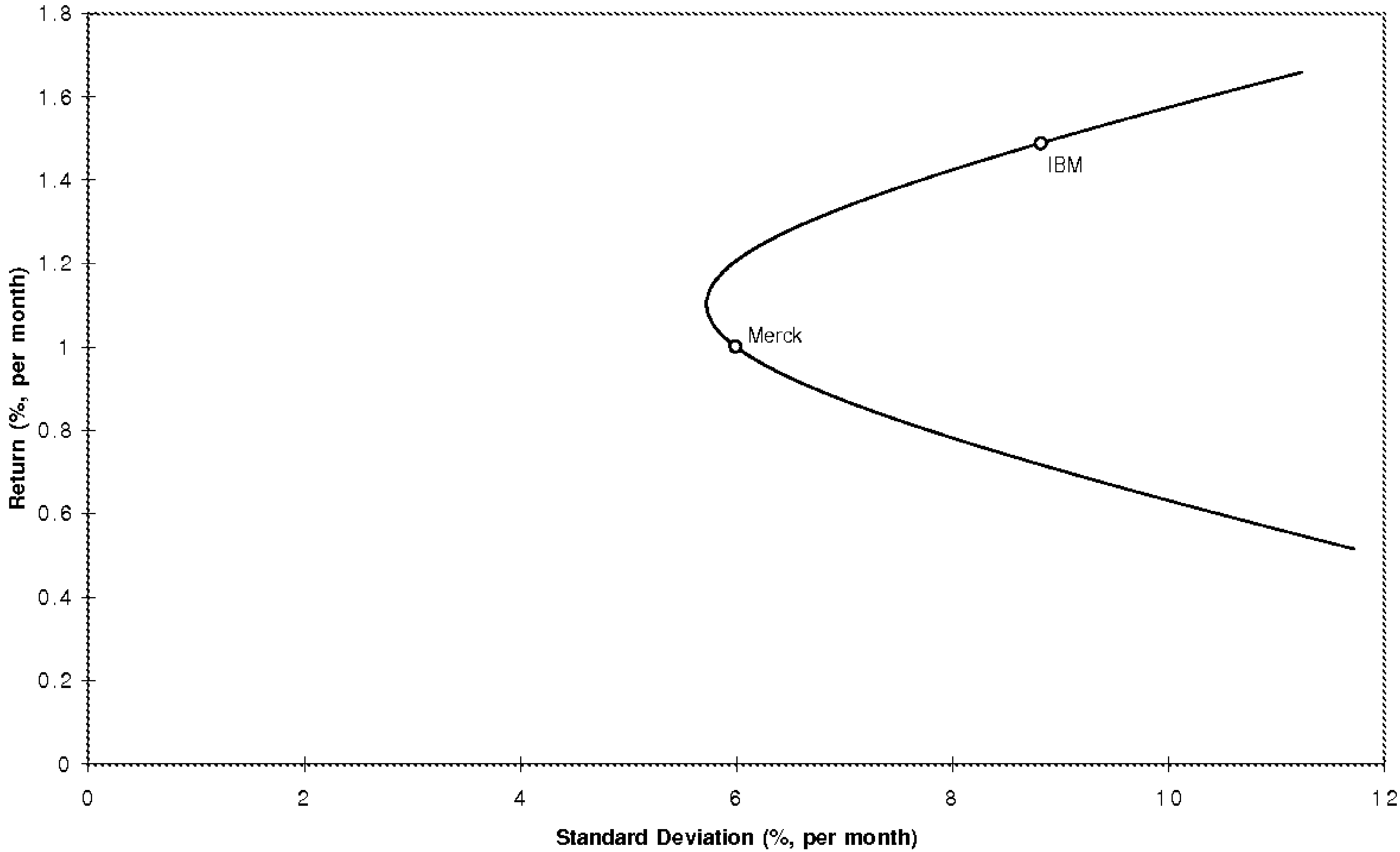
Example (ctd).

Covariances	$\tilde{r}_{IBM}$	$\tilde{r}_{Merck}$
$\tilde{r}_{IBM}$	0.007770	0.002095
$\tilde{r}_{Merck}$	0.002095	0.003587
Mean	1.49%	1.00%
SD	8.81%	5.99%

Portfolios of IBM and Merck

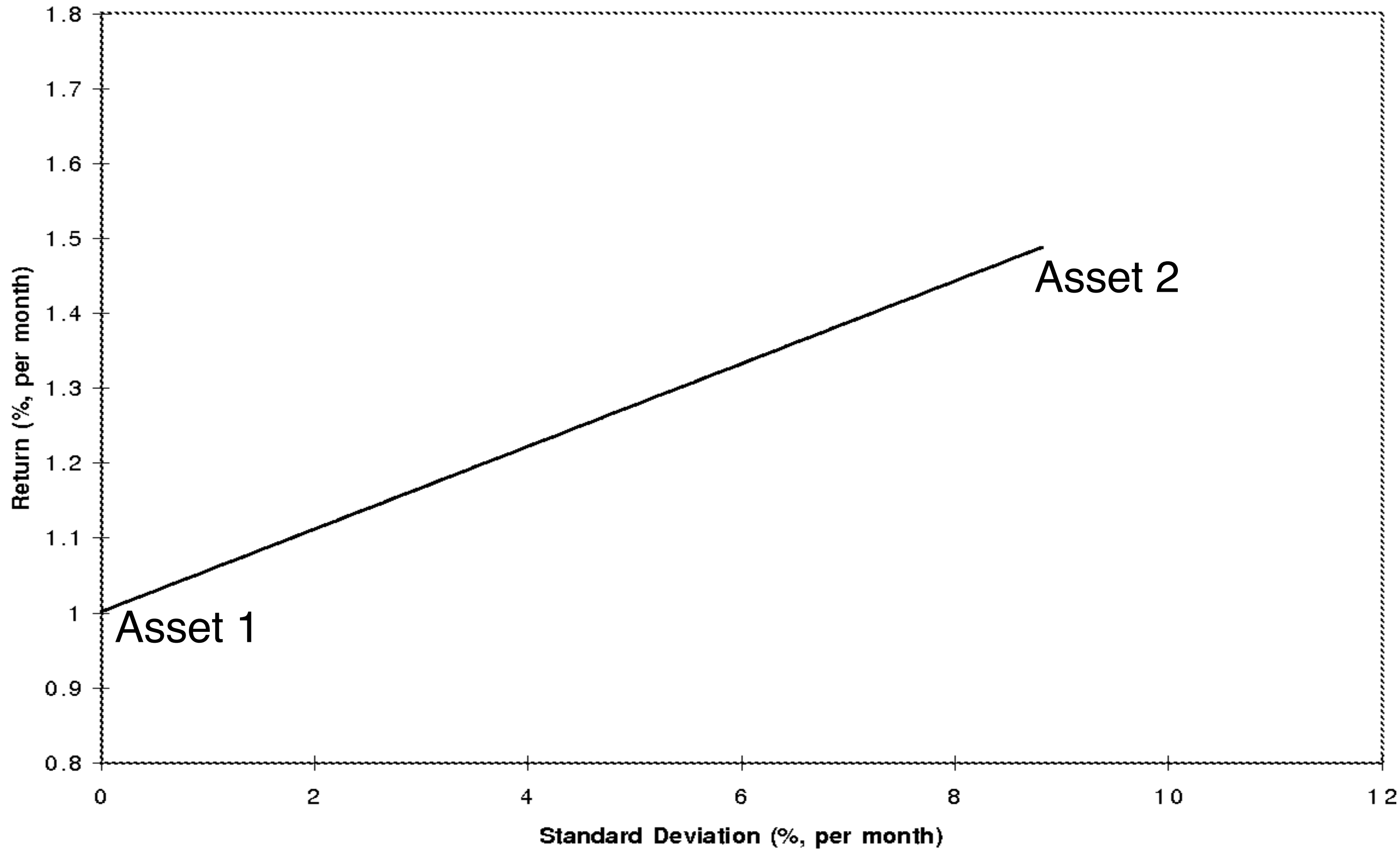
Weight in IBM (%)	-40	-20	0	20	40	60	80	100	120	140
Mean return (%)	0.80	0.90	1.00	1.10	1.20	1.29	1.39	1.49	1.59	1.69
SD (%)	7.70	6.69	5.99	5.72	5.95	6.62	7.61	8.81	10.16	11.60

Portfolio frontier when short sales are allowed

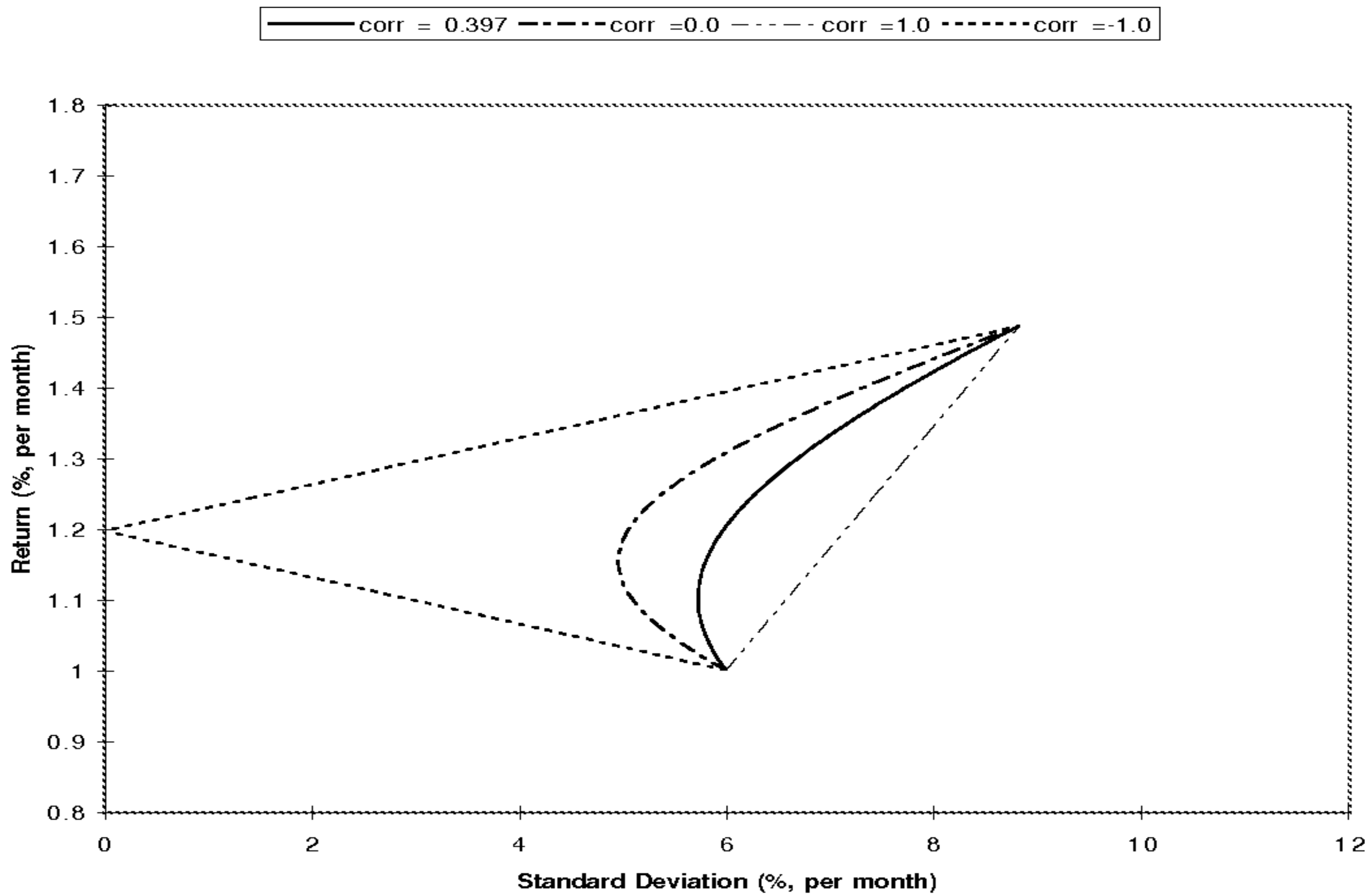


Special situations (without short sales)

Asset 1 is risk-free:

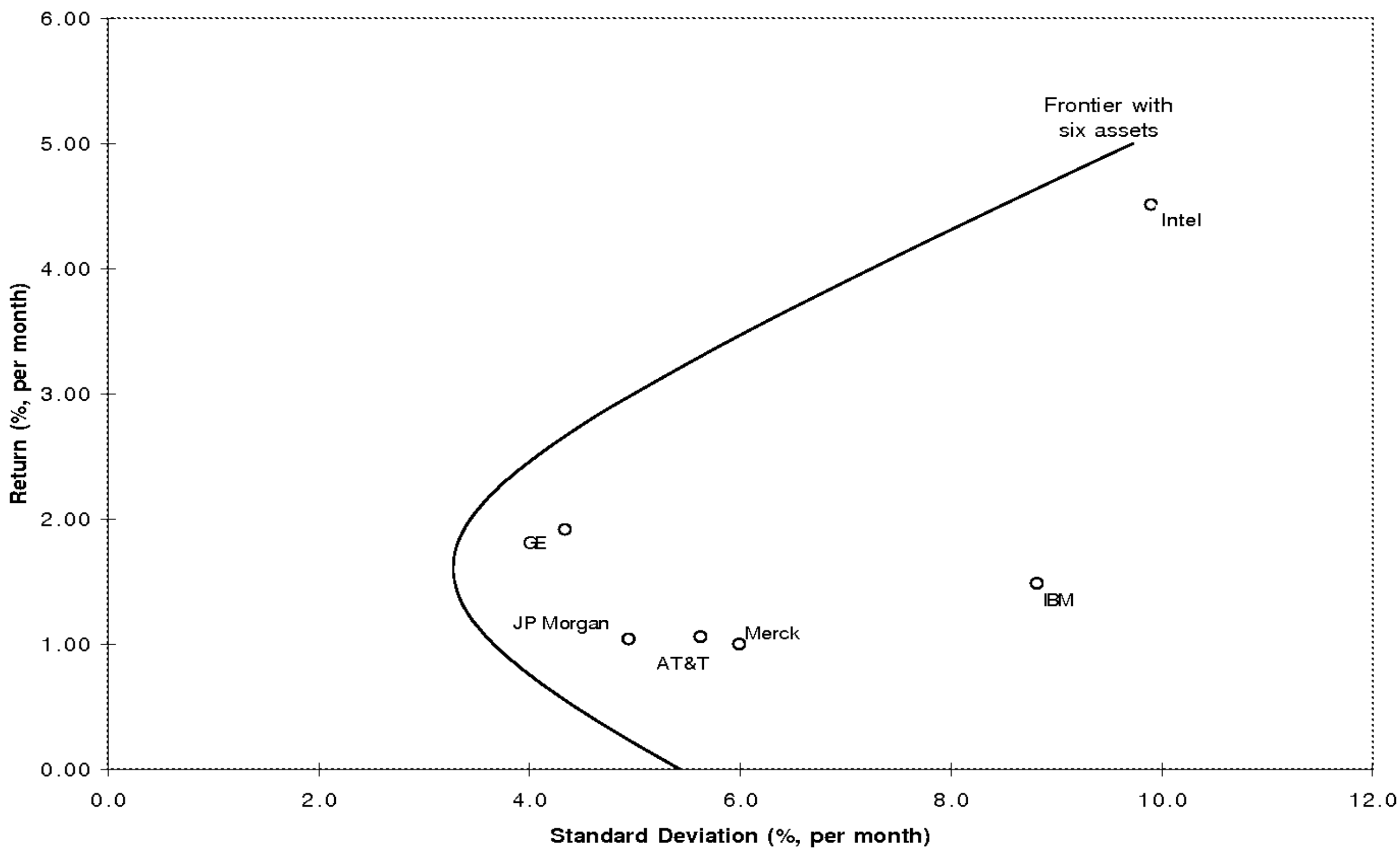


2. Perfect correlation between two assets ($\rho_{12} = \pm 1$):



Solving optimal portfolios "graphically":

Portfolio frontier from stocks of IBM, Merck, Intel, AT&T, JP Morgan & GE



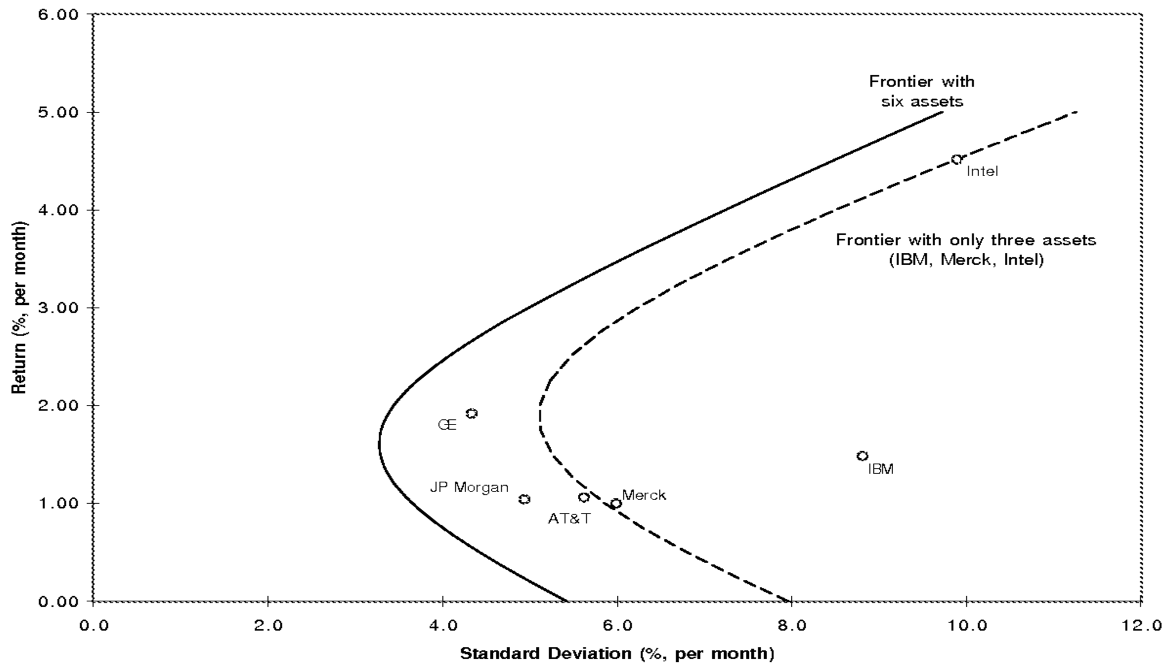
Given an expected return, the portfolio that minimizes risk (measured by StD or variance) is a mean-variance frontier portfolio.

The locus of all frontier portfolios in the mean-StD plane is called portfolio frontier.

The upper part of the portfolio frontier gives the efficient frontier portfolios.

To obtain the efficient portfolios, we need to solve the constrained optimization problem (P). A numerical solution can be found with Excel's Solver.

Portfolio frontier of IBM, Merck, Intel, AT&T, JP Morgan and GE



When more assets are included, the portfolio frontier improves, i.e., moves toward upper-left: higher mean returns and lower risk.

Intuition: Since one can always choose to ignore the new assets, including them cannot make one worse off.

When there exists a safe (risk-free) asset, each portfolio consists of the risk-free asset and risky assets.

Observation: A portfolio of risk-free and risky assets can be viewed as a portfolio of two portfolios:

1) the risk-free asset, and
2) a portfolio of only risky assets.

Example. Consider a portfolio with $40 invested in the risk-free asset and $30 each in two risky assets, IBM and Merck:

- $w_0 = 40\%$ in the risk-free asset
- $w_1 = 30\%$ in IBM and
- $w_2 = 30\%$ in Merck.

We can also view the portfolio as follows:

1) 1 - x = 40% in the risk-free asset

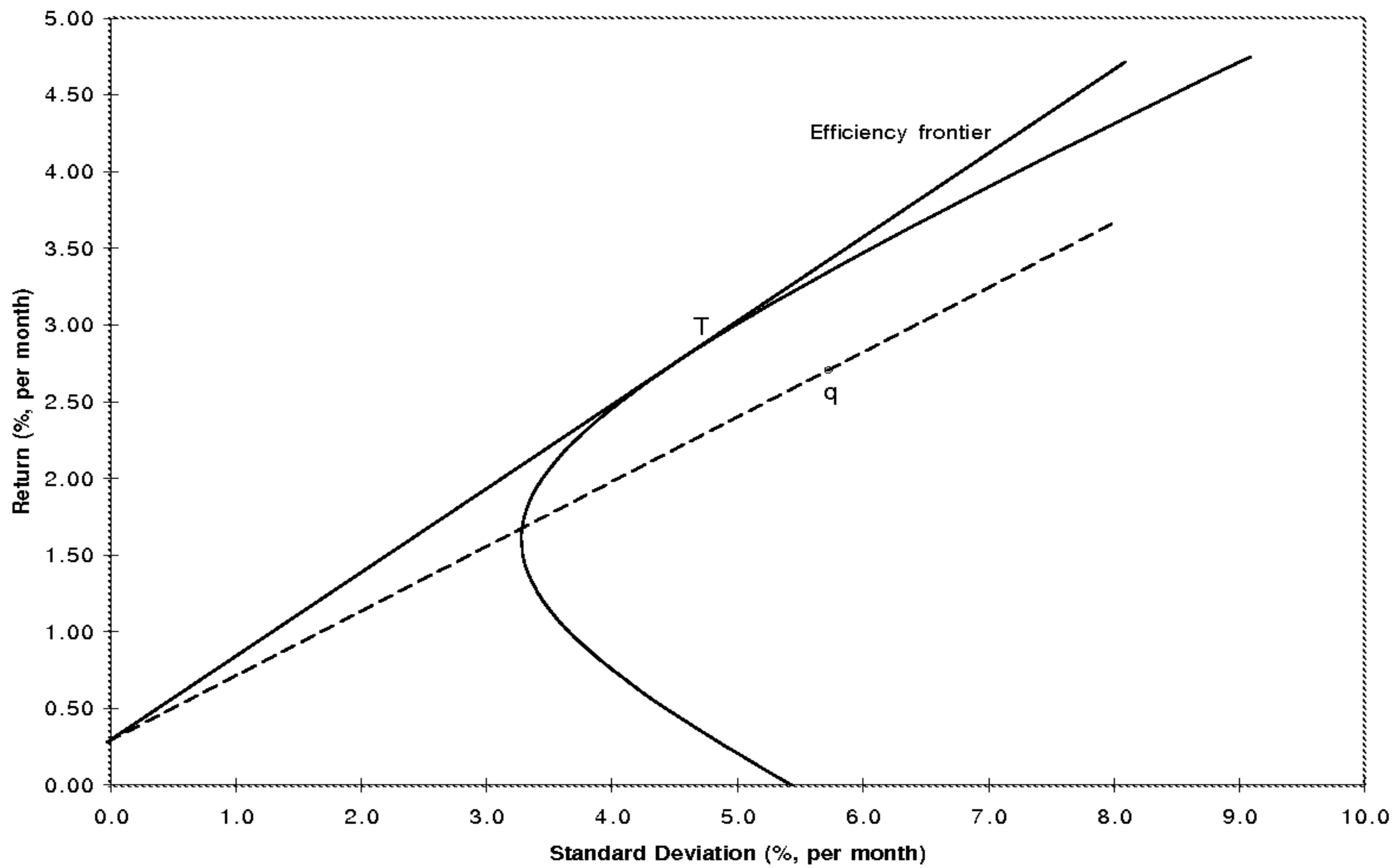
2) x = 60% in a portfolio of only risky assets which has

   a) 50% in IBM

   b) 50% in Merck.

Consider a portfolio p with x invested in a risky portfolio q, and 1-x invested in the risk-free asset. Then,

$$\begin{aligned} \bar{r}_p &= (1-x) r_F + x \bar{r}_q \\ \sigma_p^2 &= x^2 \sigma_q^2 \end{aligned}$$



With a risk-free asset, frontier portfolios are combinations of:

1) the risk-free asset

2) the tangent portfolio (consisted of only risky assets).

Sharpe ratio

A measure of a portfolio's risk-return trade-off, equal to the portfolio's risk premium divided by its volatility:

$$\boxed{\text{Sharpe Ratio} \equiv \frac{\bar{r}_p - r_F}{\sigma_p} \quad \textbf{(higher is better!)}}$$

The tangency portfolio has the highest possible Sharpe ratio of any portfolio. Like all the portfolios on the CML.

1. Risk comes in two types:
   – Diversifiable (non-systematic)
   – Non-diversifiable (systematic)
2. Diversification reduces (diversifiable) risk.
3. Investors hold frontier portfolios.
   – Large asset base improves the portfolio frontier.
4. When there is risk-free asset, frontier portfolios are linear combinations of
   – the risk-free asset, and
   – the tangent portfolio.

_ Portfolios

_ Portfolio returns

_ Diversification

_ Systematic vs. non-systematic risks

_ Optimal portfolio choices

_ Sharpe ratio