Key concepts

- Portfolios
- Portfolio returns
- Diversification
- Systematic vs. non-systematic risks
- Optimal portfolio choices
- Sharpe ratio

Readings:

- Brealey, Myers and Allen, Chapter 8, 9.1
- Bodie, Kane and Markus, Chapters 6.2, 7, 8
**What is a portfolio?**

A **portfolio** is simply a collection of assets:

- $n$ assets, each with share price $P_i$ ($i = 1, 2, \ldots, n$)
- A portfolio is a collection of $N_i$ shares of each asset $i$
- Total value of portfolio:

\[ V = N_1 P_1 + N_2 P_2 + \cdots + N_n P_n = \sum_{i=1}^{n} N_i P_i \]

- A typical portfolio has $V > 0$. Define portfolio weights:

\[ w_i = \frac{N_i P_i}{N_1 P_1 + N_2 P_2 + \cdots + N_n P_n} = \frac{N_i P_i}{V} \]

- A portfolio can then also be defined by its asset weights

\[ \{w_1, w_2, \ldots, w_n\}, \quad w_1 + w_2 + \cdots + w_n = 1 \]
Example. Your investment account of $100,000 consists of three stocks: 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C. Your portfolio is summarized by the following weights:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shares</th>
<th>Price/Share</th>
<th>Dollar Investment</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>200</td>
<td>$50</td>
<td>$10,000</td>
<td>10%</td>
</tr>
<tr>
<td>B</td>
<td>1,000</td>
<td>$60</td>
<td>$60,000</td>
<td>60%</td>
</tr>
<tr>
<td>C</td>
<td>750</td>
<td>$40</td>
<td>$30,000</td>
<td>30%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>$100,000</td>
<td>100%</td>
</tr>
</tbody>
</table>
Example (cont). Your broker informs you that you only need to keep $50,000 in your investment account to support the same portfolio of 200 shares of stock A, 1,000 shares of stock B, and 750 shares of stock C; in other words, you can buy these stocks on margin. You withdraw $50,000 to use for other purposes, leaving $50,000 in the account. Your portfolio is summarized by the following weights:

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shares</th>
<th>Price/Share</th>
<th>Dollar Investment</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>200</td>
<td>$50</td>
<td>$10,000</td>
<td>20%</td>
</tr>
<tr>
<td>B</td>
<td>1,000</td>
<td>$60</td>
<td>$60,000</td>
<td>120%</td>
</tr>
<tr>
<td>C</td>
<td>750</td>
<td>$40</td>
<td>$30,000</td>
<td>60%</td>
</tr>
<tr>
<td>Riskless Bond</td>
<td>−$50,000</td>
<td>$1</td>
<td>−$50,000</td>
<td>−100%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>$50,000</td>
<td>100%</td>
</tr>
</tbody>
</table>
**Example.** You decide to purchase a home that costs $500,000 by paying 20% of the purchase price and getting a mortgage for the remaining 80%. What are your portfolio weights for this investment?

<table>
<thead>
<tr>
<th>Asset</th>
<th>Shares</th>
<th>Price/Share</th>
<th>Dollar Investment</th>
<th>Portfolio Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home</td>
<td>1</td>
<td>$500,000</td>
<td>$500,000</td>
<td>500%</td>
</tr>
<tr>
<td>Mortgage</td>
<td>1</td>
<td>−$400,000</td>
<td>−$400,000</td>
<td>−400%</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>$100,000</td>
<td>100%</td>
</tr>
</tbody>
</table>

Leverage ratio = asset/net investment = 5

What happens to your total assets if your home price declines by 15%?
Why not pick the best asset instead of forming a portfolio?
- We don’t know which stock is best!
- Portfolios provide diversification, reducing unnecessary risks
- Portfolios can enhance performance by focusing bets
- Portfolios can customize and manage risk/reward trade-offs

How do we chose a “good” portfolio?
- What does “good” mean?
- What characteristics of a portfolio do we care about?
  - risk and reward (expected return)
  - higher expected returns are preferred
  - higher risks are not preferred
A portfolio’s characteristics are determined by the returns of its assets and its weights in them.

_ Mean returns:

<table>
<thead>
<tr>
<th>Asset</th>
<th>1</th>
<th>2</th>
<th>…</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Return</td>
<td>( \bar{r}_1 )</td>
<td>( \bar{r}_2 )</td>
<td>…</td>
<td>( \bar{r}_n )</td>
</tr>
</tbody>
</table>

_ Variances and co-variances:

<table>
<thead>
<tr>
<th></th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>…</th>
<th>( r_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 )</td>
<td>( \sigma_1^2 )</td>
<td>( \sigma_{12} )</td>
<td>…</td>
<td>( \sigma_{1n} )</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>( \sigma_{21} )</td>
<td>( \sigma_2^2 )</td>
<td>…</td>
<td>( \sigma_{2n} )</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
<tr>
<td>( r_n )</td>
<td>( \sigma_{n1} )</td>
<td>( \sigma_{n2} )</td>
<td>…</td>
<td>( \sigma_n^2 )</td>
</tr>
</tbody>
</table>

Covariance of an asset with itself is its variance:
Example. Monthly stock returns on IBM ($r_1$) and Merck ($r_2$):

\[
\begin{array}{c|c|c}
\text{Mean returns} & \tilde{r}_1 & \tilde{r}_2 \\
\hline
\tilde{r}_1 & 0.0149 \\
\tilde{r}_2 & 0.0100 \\
\end{array}
\quad
\begin{array}{c|c|c}
\text{Covariance matrix} & \tilde{r}_1 & \tilde{r}_2 \\
\hline
\tilde{r}_1 & 0.007770 & 0.002095 \\
\tilde{r}_2 & 0.002095 & 0.003587 \\
\end{array}
\]

Note: $\sigma_1 = 8.81\%$, $\sigma_2 = 5.99$ and $\rho_{12} = 0.40$. 
The portfolio return is a weighted average of the individual returns:

$$\tilde{r}_p = w_1\tilde{r}_1 + w_2\tilde{r}_2$$

**Example.** Suppose you invest $600 in IBM and $400 in Merck for a month. If the realized return is 2.5% on IBM and 1.5% on Merck over the month, what is the return on your total portfolio?

The portfolio weights are

$$w_{IBM} = \frac{600}{1000} = 60\% \quad w_{Merck} = \frac{400}{1000} = 40\%$$

$$r_p = \frac{(600)(0.025) + (400)(0.015)}{1000}$$

$$= (0.6)(0.025) + (0.4)(0.015)$$

$$= 2.1\%$$
Expected return on a portfolio with two assets

Expected portfolio return:

\[ \tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 \]

Unexpected portfolio return:

\[ \tilde{r}_p - \tilde{r}_p = w_1 (\tilde{r}_1 - \tilde{r}_1) + w_2 (\tilde{r}_2 - \tilde{r}_2) \]

Variance of return on a portfolio with two assets

The variance of the portfolio return:

\[ \sigma^2_p = w_1^2 \sigma^2_1 + w_2^2 \sigma^2_2 + 2w_1 w_2 \sigma_{12} \]

which is also the sum of all entries of the following table

<table>
<thead>
<tr>
<th></th>
<th>( w_1 \tilde{r}_1 )</th>
<th>( w_2 \tilde{r}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 \tilde{r}_1 )</td>
<td>( w_1^2 \sigma^2_1 )</td>
<td>( w_1 \sigma_{12} )</td>
</tr>
<tr>
<td>( w_2 \tilde{r}_2 )</td>
<td>( w_1 \sigma_{12} )</td>
<td>( w_2^2 \sigma^2_2 )</td>
</tr>
</tbody>
</table>
**Example.** Consider again investing in IBM and Merck stocks.

<table>
<thead>
<tr>
<th>Mean returns</th>
<th>Covariance matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{r}_1$</td>
<td>$\tilde{r}_1$</td>
</tr>
<tr>
<td>0.0149</td>
<td>0.002095</td>
</tr>
<tr>
<td>0.0100</td>
<td>0.003587</td>
</tr>
</tbody>
</table>

Consider the equally weighted portfolio: $w_1 = w_2 = 0.5$

Mean of portfolio return: $\tilde{r}_p = (0.5)(0.0149) + (0.5)(0.0100) = 1.25\%$

Variance of portfolio return:

\[
\sigma_p^2 = (0.5)^2(0.007770) + (0.5)^2(0.003587) + (2)(0.5)^2(0.002095)
\]
\[
= 0.003888
\]
\[
\sigma_p = 6.23\%
\]
_ Expected portfolio return:

\[
\tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + w_3 \tilde{r}_3
\]

_ The variance of the portfolio return:

\[
\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2w_1 w_2 \sigma_{12} + 2w_1 w_3 \sigma_{13} + 2w_2 w_3 \sigma_{23}
\]

**Example.** IBM, Merck and Intel returns have covariance matrix:

<table>
<thead>
<tr>
<th></th>
<th>(\tilde{r}_{IBM})</th>
<th>(\tilde{r}_{Merck})</th>
<th>(\tilde{r}_{Intel})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tilde{r}_{IBM})</td>
<td>0.007770</td>
<td>0.002095</td>
<td>0.001189</td>
</tr>
<tr>
<td>(\tilde{r}_{Merck})</td>
<td>0.002095</td>
<td>0.003587</td>
<td>0.000229</td>
</tr>
<tr>
<td>(\tilde{r}_{Intel})</td>
<td>0.001189</td>
<td>0.000229</td>
<td>0.009790</td>
</tr>
</tbody>
</table>

_ What is the risk (StD) of the equally weighted portfolio?

\[
\sigma_p^2 = \left(\frac{1}{3}\right)^2 \times (\text{Sum of all entries of covariance matrix}) = 0.003130
\]

\[
\sigma_p = 5.59\%
\]

_ For individual assets: \(\sigma_{IBM} = 8.81\%\), \(\sigma_{Merck} = 5.99\%\), \(\sigma_{Intel} = 9.89\%\)
We now consider a portfolio of $n$ assets:

$$\{w_1, w_2, \ldots, w_n\}, \quad \sum_i w_i = 1$$

1. The return on the portfolio is:

$$\tilde{r}_p = w_1 \tilde{r}_1 + w_2 \tilde{r}_2 + \cdots + w_n \tilde{r}_n$$

2. The expected return on the portfolio is:

$$\bar{r}_p = \mathbb{E}[r_p] = w_1 \bar{r}_1 + w_2 \bar{r}_2 + \cdots + w_n \bar{r}_n$$

3. The variance of portfolio return is:

$$\sigma_p^2 = \text{Var}[\tilde{r}_p] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}, \quad \sigma_{ii} = \sigma_i^2$$

4. The volatility (StD) of portfolio return is:

$$\sigma_p = \sqrt{\text{Var}[\tilde{r}_p]} = \sqrt{\sigma_p^2}$$
The variance of portfolio return can be computed by summing up all the entries to the following table:

\[
\begin{array}{cccc}
  & w_1 r_1 & w_2 r_2 & \cdots & w_n r_n \\
 w_1 r_1 & w_1^2 \sigma_1^2 & w_1 w_2 \sigma_{12} & \cdots & w_1 w_n \sigma_{1n} \\
 w_2 r_2 & w_2 w_1 \sigma_{21} & w_2^2 \sigma_2^2 & \cdots & w_2 w_n \sigma_{2n} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 w_n r_n & w_n w_1 \sigma_{n1} & w_n w_2 \sigma_{n2} & \cdots & w_n^2 \sigma_n^2 \\
\end{array}
\]

The variance of a sum is not just the sum of variances! We also need to account for the covariances. In order to calculate return variance of a portfolio, we need:

a) portfolio weights  
b) individual variances  
c) all covariances
Diversification reduces risk.

1. Two assets:
**Example.** Two assets with the same annual return StD of 35%. Consider a portfolio p with weight w in asset 1 and 1-w in asset 2.

\[ \sigma_p = \sqrt{w^2 \sigma_1^2 + (1-w)^2 \sigma_2^2 + 2w(1-w)\sigma_{12}}. \]

StD of portfolio return is less than the StD of each individual asset.
2. Multiple assets:
Certain risks cannot be diversified away.

Impact of diversification on portfolio risk

Risk comes in two types:

- Diversifiable risk
- Non-diversifiable (market, systematic) risk due to macro (business cycle, inflation, etc.) / market conditions (liquidity)
**Example.** An equally-weighted portfolio of n assets:

\[
\begin{array}{c|ccc}
& w_1 r_1 & \cdots & w_n r_n \\
\hline
w_1 r_1 & w_1^2 \sigma_1^2 & \cdots & w_1 w_n \sigma_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
w_n r_n & w_n w_1 \sigma_{n1} & \cdots & w_n^2 \sigma_n^2 \\
\end{array}
\]

- A typical variance term: \( \left( \frac{1}{n} \right)^2 \sigma_{ii} \).
  - There are n variance terms.
- A typical covariance term: \( \left( \frac{1}{n} \right)^2 \sigma_{ij}, (i \neq j) \).
  - There are \( n^2 - n \) covariance terms.
Add all the terms:

\[
\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} \left( \frac{1}{n} \right)^2 \sigma_{ii} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} \left( \frac{1}{n} \right)^2 \sigma_{ij}
\]

\[
= \left( \frac{1}{n} \right) \left( \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 \right) + \left( \frac{n^2 - n}{n^2} \right) \left( \frac{1}{n^2 - n} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sigma_{ij} \right)
\]

\[
= \left( \frac{1}{n} \right) \text{(average variance)} + \left( \frac{n^2 - n}{n^2} \right) \text{(average covariance)}
\]

As n becomes very large:

_ Contribution of variance terms goes to zero.
_ Contribution of covariance terms goes to “average covariance”.
Example (ctd). The average stock has a monthly standard deviation of 10% and the average correlation between stocks is 0.4. If you invest the same amount in each stock, what is the variance of the portfolio?

\[
\begin{align*}
\text{Cov}[R_i, R_j] &= \rho_{ij} \sigma_i \sigma_j = 0.40 \times 0.10 \times 0.10 = 0.004 \\
\text{Var}[R_p] &= \frac{1}{n} 0.10^2 + \frac{n-1}{n} 0.004 \approx 0.004 \quad \text{if } n \text{ large} \\
\sigma_p &\approx \sqrt{0.004} = 6.3\%
\end{align*}
\]

What is the correlation is 0? What if it is 1?
Example (ctd).
How to choose a portfolio:

- Minimize risk for a given expected return? or
- Maximize expected return for a given risk?
Formally, we need to solve the following problem:

\[(P) \quad \text{Minimize} \quad \left\{ w_1, \ldots, w_n \right\} \quad \text{subject to} \quad \begin{align*}
\sigma_p^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
\sum_{i=1}^{n} w_i &= 1 \\
\sum_{i=1}^{n} w_i \bar{r}_i &= \bar{r}_p
\end{align*}\]
Without short sales

Example (ctd). IBM and Merck:

\[
\bar{r}_p = w\bar{r}_1 + (1-w)\bar{r}_2 \\
\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}
\]
Portfolio frontier when short sales are not allowed
With short sales

When short sales are allowed, portfolio weights are unrestricted.

Example (ctd).

<table>
<thead>
<tr>
<th>Covariances</th>
<th>$\tilde{r}_{\text{IBM}}$</th>
<th>$\tilde{r}_{\text{Merck}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{r}_{\text{IBM}}$</td>
<td>0.007770</td>
<td>0.002095</td>
</tr>
<tr>
<td>$\tilde{r}_{\text{Merck}}$</td>
<td>0.002095</td>
<td>0.003587</td>
</tr>
<tr>
<td>Mean</td>
<td>1.49%</td>
<td>1.00%</td>
</tr>
<tr>
<td>SD</td>
<td>8.81%</td>
<td>5.99%</td>
</tr>
</tbody>
</table>

Portfolios of IBM and Merck

<table>
<thead>
<tr>
<th>Weight in IBM (%)</th>
<th>-40</th>
<th>-20</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>140</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean return (%)</td>
<td>0.80</td>
<td>0.90</td>
<td>1.00</td>
<td>1.10</td>
<td>1.20</td>
<td>1.29</td>
<td>1.39</td>
<td>1.49</td>
<td>1.59</td>
<td>1.69</td>
</tr>
<tr>
<td>SD (%)</td>
<td>7.70</td>
<td>6.69</td>
<td>5.99</td>
<td>5.72</td>
<td>5.95</td>
<td>6.62</td>
<td>7.61</td>
<td>8.81</td>
<td>10.16</td>
<td>11.60</td>
</tr>
</tbody>
</table>
Portfolio frontier when short sales are allowed

Return (% per month)

Standard Deviation (% per month)
Special situations (without short sales)

Asset 1 is risk-free:

Return (% per month)

Standard Deviation (% per month)
2. Perfect correlation between two assets ($\rho_{12} = \pm 1$):

![Graph showing the effect of correlation on portfolio returns and standard deviations]
Solving optimal portfolios “graphically”:

Portfolio frontier from stocks of IBM, Merck, Intel, AT&T, JP Morgan & GE
Given an expected return, the portfolio that minimizes risk (measured by StD or variance) is a \textit{mean-variance frontier portfolio}. The locus of all frontier portfolios in the mean-StD plane is called \textit{portfolio frontier}. The upper part of the portfolio frontier gives the \textit{efficient frontier portfolios}.

To obtain the efficient portfolios, we need to solve the constrained optimization problem (P). A numerical solution can be found with Excel’s Solver.
When more assets are included, the portfolio frontier improves, i.e., moves toward upper-left: higher mean returns and lower risk.

Intuition: Since one can always choose to ignore the new assets, including them cannot make one worse off.
When there exists a safe (risk-free) asset, each portfolio consists of the risk-free asset and risky assets.

Observation: A portfolio of risk-free and risky assets can be viewed as a portfolio of two portfolios:

1) the risk-free asset, and
2) a portfolio of only risky assets.

Example. Consider a portfolio with $40 invested in the risk-free asset and $30 each in two risky assets, IBM and Merck:

- \( w_0 = 40\% \) in the risk-free asset
- \( w_1 = 30\% \) in IBM and
- \( w_2 = 30\% \) in Merck.
We can also view the portfolio as follows:

1) \(1 - x = 40\%\) in the risk-free asset

2) \(x = 60\%\) in a portfolio of only risky assets which has
   a) 50% in IBM
   b) 50% in Merck.

Consider a portfolio \(p\) with \(x\) invested in a risky portfolio \(q\), and \(1-x\) invested in the risk-free asset. Then,

\[
\bar{r}_p = (1-x)r_F + x\bar{r}_q
\]

\[
\sigma^2_p = x^2\sigma^2_q
\]
With a risk-free asset, frontier portfolios are combinations of:

1) the risk-free asset

2) the tangent portfolio (consisted of only risky assets).
Sharpe ratio

A measure of a portfolio’s risk-return trade-off, equal to the portfolio’s risk premium divided by its volatility:

\[
\text{Sharpe Ratio} \equiv \frac{\bar{r}_p - r_F}{\sigma_p} \quad (\text{higher is better!})
\]

The tangency portfolio has the highest possible Sharpe ratio of any portfolio. Like all the portfolios on the CML.
1. Risk comes in two types:
   - Diversifiable (non-systematic)
   - Non-diversifiable (systematic)

2. Diversification reduces (diversifiable) risk.

3. Investors hold frontier portfolios.
   - Large asset base improves the portfolio frontier.

4. When there is risk-free asset, frontier portfolios are linear combinations of
   - the risk-free asset, and
   - the tangent portfolio.
Key concepts

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- Diversification
- Systematic vs. non-systematic risks
- Optimal portfolio choices
- Sharpe ratio