

Resource Allocation in Multiple Access Channels

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Abstract—We consider the problem of rate allocation in a Gaussian multiple-access channel, with the goal of maximizing a utility function over transmission rates. In contrast to the literature which focuses on linear utility functions, we study general concave utility functions. We present a gradient projection algorithm for this problem. Since the constraint set of the problem is described by exponentially many constraints, methods that use exact projections are computationally intractable. Therefore, we develop a new method that uses approximate projections. We use the polymatroid structure of the capacity region to show that the approximate projection can be implemented by a recursive algorithm in time polynomial in the number of users. We further propose another algorithm for implementing the approximate projections using rate-splitting and show improved bounds on its convergence time.

I. INTRODUCTION

Dynamic allocation of communication resources such as bandwidth or transmission power is a central issue in multiple access channels in view of the time varying nature of the channel and interference effects. Most of the existing literature on resource allocation in multiple access channels focuses on specific communication schemes such as TDMA (time-division multiple access) [1] and CDMA (code-division multiple access) [2], [3] systems. An exception is the work by Tse *et al.* [4], who introduced the notion of *throughput capacity* for the fading channel with Channel State Information (CSI) and studied dynamic rate allocation policies with the goal of maximizing a linear utility function of rates over the throughput capacity region.

In this paper, we consider the problem of rate allocation in a multiple access channel with perfect CSI. Contrary to the linear case in [4], we consider maximizing a general utility function of transmission rates over the capacity region. General concave utility functions allow us to model different performance metrics and fairness criteria (cf. Shenker [5], Srikant [6]). In view of space restrictions, we focus on the non-fading channel in this paper. In our companion paper [7], we extend our analysis to the fading channel.

Our contributions can be summarized as follows.

We introduce a gradient projection method for the problem of maximizing a concave utility function of rates over the capacity

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region of a non-fading channel. We establish the convergence of the method to the optimal solution of the problem. Since the capacity region of the multiple-access channel is described by a number of constraints exponential in the number of users, the projection operation used in the method can be computationally expensive. To reduce the computational complexity, we introduce a new method that uses *approximate projections*. By exploiting the polymatroid structure of the capacity region, we show that the approximate projection operation can be implemented in polynomial time using submodular function minimization algorithms. Moreover, we present a more efficient algorithm for the approximate projection problem which relies on rate-splitting [8]. This algorithm also provides the extra information that allows the receiver to decode the message by successive cancellation.

Other than the papers cited above, our work is also related to the work of Vishwanath *et al.* [9] which builds on [4] and takes a similar approach to the resource allocation problem for linear utility functions. Other works address different criteria for resource allocation including minimizing the weighted sum of transmission powers [10], and considering Quality of Service (QoS) constraints [11]. In contrast to this literature, we consider the utility maximization framework for general concave utility functions.

The remainder of this paper is organized as follows: In Section II, we introduce the model and describe the capacity region of a multiple-access channel. In Section III, we consider the utility maximization problem in non-fading channel and present the gradient projection method. In Section IV, we address the complexity of the projection problem. Finally, we give our concluding remarks in Section V.

Regarding the notation, we denote by x_i the i -th component of a vector \mathbf{x} . We denote the nonnegative orthant by \mathbb{R}_+^n , i.e., $\mathbb{R}_+^n = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq 0\}$. We write \mathbf{x}' to denote the transpose of a vector \mathbf{x} . We use $\|\mathbf{x}\|$ to denote the standard Euclidean norm, $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}}$, and $\mathcal{P}(\bar{\mathbf{x}})$ to denote the exact projection of a vector $\bar{\mathbf{x}} \in \mathbb{R}^n$ on a nonempty closed convex set X , i.e.,

$$\mathcal{P}(\bar{\mathbf{x}}) = \arg \min_{\mathbf{x} \in X} \|\bar{\mathbf{x}} - \mathbf{x}\|.$$

II. SYSTEM MODEL

We consider M users sharing the same media to communicate to a single receiver. We model the channel as a Gaussian multiple access channel with flat fading effects

$$Y(n) = \sum_{i=1}^M \sqrt{H_i(n)} X_i(n) + Z(n), \quad (1)$$

where $X_i(n)$ are the transmitted waveform with average power P_i , $H_i(n)$ is the channel gain corresponding to the i -th user and

$Z(n)$ is white Gaussian noise with variance N_0 . We assume that the channel gains are known to all users and the receiver ¹.

We focus on the non-fading case when the channel gains are fixed. We assume without loss of generality that all channel gains are equal to unity. The capacity region of the Gaussian multiple-access channel is described as follows [12]:

$$C_g(\mathbf{P}) = \left\{ \mathbf{R} \in \mathbb{R}_+^M : \sum_{i \in S} R_i \leq C\left(\sum_{i \in S} P_i, N_0\right), \right. \\ \left. \text{for all } S \subseteq \mathcal{M} = \{1, \dots, M\} \right\}, \quad (2)$$

where P_i and R_i are the i -th user's power and rate, respectively. $C(P, N)$ denotes Shannon's formula for the capacity of an AWGN channel given by

$$C(P, N) = \frac{1}{2} \log\left(1 + \frac{P}{N}\right) \text{ nats.} \quad (3)$$

III. RESOURCE ALLOCATION IN NON-FADING CHANNEL

Consider the following utility maximization problem in a M -user non-fading multiple-access channel with channel gains fixed to unity.

$$\begin{aligned} & \text{maximize} && u(\mathbf{R}) \\ & \text{subject to} && \mathbf{R} \in C_g(\mathbf{P}), \end{aligned} \quad (4)$$

where R_i and P_i are i -th user rate and power, respectively. The utility function $u(\mathbf{R})$ is assumed to satisfy the following conditions.

Assumption 1:

- (a) The utility function $u(\mathbf{R})$ is concave with respect to vector \mathbf{R} .
- (b) The utility function $u(\mathbf{R})$ is monotonically non-decreasing with respect to R_i , for $i = 1, \dots, M$.
- (c) There exists a scalar B such that

$$\|\mathbf{g}\| \leq B, \quad \text{for all } \mathbf{g} \in \partial u(\mathbf{R}),$$

where $\partial u(\mathbf{R})$ denotes the subdifferential of u at \mathbf{R} .

The maximization problem in (4) is a convex program and the optimal solution can be obtained by several variational methods such as the gradient projection method. The gradient projection method with exact projection is typically user for problems where the projection operation is simple, i.e., for problems with simple constraint sets such as the non-negative orthant or a simplex. However, the constraint set in (4) is defined by exponentially many constraints, making the projection problem computationally intractable. To alleviate this problem, we use an approximate projection, which is obtained by successively projecting on some violated constraint.

Definition 1: Let $X = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$ where A has non-negative entries. Let $\mathbf{y} \in \mathbb{R}^n$ violate the constraint $\mathbf{a}'_i \mathbf{x} \leq b_i$, for $i \in \{i_1, \dots, i_l\}$. The approximate projection of \mathbf{y} on X , denoted by $\tilde{\mathcal{P}}$, is given by

$$\tilde{\mathcal{P}}(\mathbf{y}) = \mathcal{P}_{i_1}(\dots(\mathcal{P}_{i_{l-1}}(\mathcal{P}_{i_l}(\mathbf{y}))),$$

¹This is assumption is satisfied in practice when the receiver measures the channels and feeds back the channel information to the users.

where \mathcal{P}_{i_k} denotes the exact projection on the hyperplane $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}'_{i_k} \mathbf{x} = b_{i_k}\}$.

Proposition 1: The approximate projection $\tilde{\mathcal{P}}$ given in Definition 1 has the following properties:

- (i) For any $\mathbf{y} \in \mathbb{R}^n$, $\tilde{\mathcal{P}}(\mathbf{y})$ is feasible with respect to set X , i.e., $\tilde{\mathcal{P}}(\mathbf{y}) \in X$.
- (ii) $\tilde{\mathcal{P}}$ is pseudo-nonexpansive, i.e.,

$$\|\tilde{\mathcal{P}}(\mathbf{y}) - \tilde{\mathbf{y}}\| \leq \|\mathbf{y} - \tilde{\mathbf{y}}\|, \quad \text{for all } \tilde{\mathbf{y}} \in X. \quad (5)$$

Proof: For part (i), it is straightforward to see that $\mathcal{P}_i(\mathbf{y})$ is given by (c.f. [13] Sec. 2.1.1)

$$\mathcal{P}_i(\mathbf{y}) = \mathbf{y} - \frac{\mathbf{a}'_i \mathbf{y} - b_i}{\|\mathbf{a}_i\|} \mathbf{a}_i.$$

Since \mathbf{a}_i just has non-negative entries, all components of \mathbf{y} are decreased after projection and hence, the constraint i will not be violated in the subsequent projections. Given an infeasible vector $\mathbf{y} \in \mathbb{R}^n$, the approximate projection operation given in Definition 1 yields a feasible vector with respect to set X .

Part (ii) can be verified using the nonexpansiveness property of projection on a convex set. Since $\tilde{\mathbf{y}}$ is a fixed point of \mathcal{P}_i for all i , we have

$$\begin{aligned} \|\tilde{\mathcal{P}}(\mathbf{y}) - \tilde{\mathbf{y}}\| &= \|\mathcal{P}_{i_1}(\dots(\mathcal{P}_{i_l}(\mathbf{y}))) - \mathcal{P}_{i_1}(\dots(\mathcal{P}_{i_l}(\tilde{\mathbf{y}})))\| \\ &\leq \|\mathcal{P}_{i_2}(\dots(\mathcal{P}_{i_l}(\mathbf{y}))) - \mathcal{P}_{i_2}(\dots(\mathcal{P}_{i_l}(\tilde{\mathbf{y}})))\| \\ &\quad \vdots \\ &\leq \|\mathbf{y} - \tilde{\mathbf{y}}\|. \end{aligned} \quad (6)$$

■

Note that the result of approximate projection depends on the order of projections on violated constraints and hence it is not unique. The k -th iteration of the gradient projection method with approximate projection is given by

$$\mathbf{R}^{k+1} = \tilde{\mathcal{P}}(\mathbf{R}^k + \alpha^k \mathbf{g}^k), \quad \mathbf{g}^k \in \partial u(\mathbf{R}^k), \quad (7)$$

where \mathbf{g}^k is a subgradient at \mathbf{R}^k , and α^k denotes the stepsize. The following theorem provides a sufficient condition that can be used to establish convergence to the optimal solution.

Theorem 1: Let Assumption 1 hold, and \mathbf{R}^* be an optimal solution of problem (4). Also, let the sequence $\{\mathbf{R}^k\}$ be generated by the iteration in (7). If the stepsize α^k satisfies

$$0 < \alpha^k < \frac{2(u(\mathbf{R}^*) - u(\mathbf{R}^k))}{\|\mathbf{g}^k\|^2}, \quad (8)$$

then

$$\|\mathbf{R}^{k+1} - \mathbf{R}^*\| < \|\mathbf{R}^k - \mathbf{R}^*\|. \quad (9)$$

Proof: We have

$$\begin{aligned} \|\mathbf{R}^k + \alpha^k \mathbf{g}^k - \mathbf{R}^*\|^2 &= \|\mathbf{R}^k - \mathbf{R}^*\|^2 + 2\alpha^k (\mathbf{R}^k - \mathbf{R}^*)' \mathbf{g}^k \\ &\quad + (\alpha^k)^2 \|\mathbf{g}^k\|^2. \end{aligned}$$

By concavity of u , we have

$$(\mathbf{R}^* - \mathbf{R}^k)' \mathbf{g}^k \geq u(\mathbf{R}^*) - u(\mathbf{R}^k). \quad (10)$$

Hence,

$$\begin{aligned} \|\mathbf{R}^k + \alpha^k \mathbf{g}^k - \mathbf{R}^*\|^2 &\leq \|\mathbf{R}^k - \mathbf{R}^*\|^2 \\ &\quad - \alpha^k \left[2 \left(u(\mathbf{R}^*) - u(\mathbf{R}^k) \right) - (\alpha^k) \|\mathbf{g}^k\|^2 \right]. \end{aligned}$$

If the stepsize satisfies (8), the above relation yields the following

$$\|\mathbf{R}^k + \alpha^k \mathbf{g}^k - \mathbf{R}^*\| < \|\mathbf{R}^k - \mathbf{R}^*\|.$$

Now by applying pseudo-nonexpansiveness of the approximate projection we have

$$\begin{aligned} \|\mathbf{R}^{k+1} - \mathbf{R}^*\| &= \|\tilde{\mathcal{P}}(\mathbf{R}^k + \alpha^k \mathbf{g}^k) - \mathbf{R}^*\| \\ &\leq \|\mathbf{R}^k + \alpha^k \mathbf{g}^k - \mathbf{R}^*\| < \|\mathbf{R}^k - \mathbf{R}^*\|. \end{aligned}$$

Proposition 2: Let Assumption 1 hold. Also, let the sequence $\{\mathbf{R}^k\}$ be generated by the iteration in (7). If the stepsize α^k satisfies (8), then $\{\mathbf{R}^k\}$ converges to an optimal solution \mathbf{R}^* .

Proof: See Proposition 8.2.7 of [14].

The convergence analysis for this method can be extended for different stepsize rules. For instance, we can employ diminishing stepsize, i.e.,

$$\alpha^k \rightarrow 0, \quad \sum_{k=0}^{\infty} \alpha^k = \infty,$$

or more complicated dynamic stepsize selection rules such as the *path-based incremental target level* algorithm proposed by Brännlund [15] which guarantees convergence to the optimal solution, and has better convergence rate compared to the diminishing stepsize rule.

IV. COMPLEXITY OF THE PROJECTION PROBLEM

Even though the approximate projection is simply obtained by successive projection on the violated constraints, it requires to find the violated constraints among exponentially many constraints describing the constraint set. In this section, we exploit the special structure of the capacity constraints so that each gradient projection step in (7) can be performed in polynomial time in M .

Definition 2: Let $f : 2^{\mathcal{M}} \rightarrow \mathbb{R}$ be a function defined over all subsets of \mathcal{M} . f is *submodular* if

$$f(S \cup T) + f(S \cap T) \leq f(S) + f(T), \quad \text{for all } S, T \in 2^{\mathcal{M}}. \quad (11)$$

Proposition 3: For any $\bar{\mathbf{R}} \in \mathbb{R}_+^M$, finding the most violated capacity constraint in (2) is equivalent to a *submodular function minimization* (SFM) problem.

Proof: Define $f_C(S) : 2^{\mathcal{M}} \rightarrow \mathbb{R}$ as follows

$$f_C(S) = C\left(\sum_{i \in S} P_i, N_0\right), \quad \text{for all } S \subseteq \mathcal{M}. \quad (12)$$

It is straightforward to see that f_C is a submodular function. We can rewrite the capacity constraints in (2) as

$$f_C(S) - \sum_{i \in S} R_i \geq 0, \quad \text{for all } S \subseteq \mathcal{M}. \quad (13)$$

Thus, the most violated constraint at $\bar{\mathbf{R}}$ is given by

$$S^* = \arg \min_{S \in 2^{\mathcal{M}}} f_C(S) - \sum_{i \in S} \bar{R}_i.$$

Since summation of a submodular and a linear function is also submodular, the problem above is of the form of submodular function minimization. ■

It is first shown by Grötschel *et al.* [16] that SFM problem can be solved in strongly polynomial time. There are several fully combinatorial strongly polynomial algorithms in the literature. The best known algorithm for SFM proposed by Orlin [17] has running time $O(M^6)$ for the submodular function defined in (12). Note that approximate projection does not require any specific order for successive projections. Hence, finding the most violated constraint is not necessary for approximate projection. In view of this fact, a more efficient algorithm based on rate-splitting is presented in Appendix I, to find a violated constraint. This algorithm runs in $O(M^2 \log M)$ time.

Although a violated constraint can be obtained in polynomial time, it does not guarantee that the approximate projection can be performed in polynomial time. Because it is possible to have exponentially many constraints violated at some point and hence the total running time of the projection would be exponential in M . However, we show that for small enough stepsize in the gradient projection iteration (7), no more than M constraints can be violated at each iteration. Let us first define the notion of expansion for a polyhedron.

Definition 3: Let Q be a polyhedron described by a set of linear constraints, i.e.,

$$Q = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}. \quad (14)$$

Define the *expansion* of Q by δ , denoted by $\mathcal{E}_\delta(Q)$, as the polyhedron obtained by relaxing all the constraints in (14), i.e., $\mathcal{E}_\delta(Q) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b} + \delta \mathbf{1}\}$, where $\mathbf{1}$ is the vector of all ones.

Lemma 1: Let f_C be as defined in (12). There exists a positive scalar δ satisfying

$$\begin{aligned} \delta &\leq \frac{1}{2} (f_C(S) + f_C(T) - f_C(S \cap T) - f_C(S \cup T)), \\ &\quad \text{for all } S, T \in 2^{\mathcal{M}}, \quad S \cap T \neq S, T, \end{aligned} \quad (15)$$

such that any point in the relaxed capacity region of an M -user multiple-access channel, $\mathcal{E}_\delta(C_g)$, violates no more than M constraints of C_g defined in (2).

Proof: Existence of a positive scalar δ satisfying (15) follows from submodularity of f_C , and the fact that neither S nor T contains the other one.

Suppose for some $\mathbf{R} \in \mathcal{E}_\delta(C_g)$, there are $M + 1$ constraints of C_g violated. There are at least two violated constraints corresponding to some sets $S, T \in 2^{\mathcal{M}}$ where $S \cap T \neq S, T$. Because it is not possible to have $M + 1$ non-empty nested sets in $2^{\mathcal{M}}$. We have

$$-\sum_{i \in S} R_i < -f_C(S), \quad (16)$$

$$-\sum_{i \in T} R_i < -f_C(T). \quad (17)$$

Since \mathbf{R} is feasible in the relaxed region,

$$\sum_{i \in S \cap T} R_i \leq f_C(S \cap T) + \delta, \quad (18)$$

$$\sum_{i \in S \cup T} R_i \leq f_C(S \cup T) + \delta. \quad (19)$$

Note that if $S \cap T = \emptyset$, (18) reduces to $0 \leq \delta$ which is a valid inequality.

By summing the above inequalities we conclude

$$\delta > \frac{1}{2}(f_C(S) + f_C(T) - f_C(S \cap T) - f_C(S \cup T)), \quad (20)$$

which is a contradiction. \blacksquare

Theorem 2: Let Assumption 1 hold. Let $P_1 \leq P_2 \leq \dots \leq P_M$ be the transmission powers.

If the stepsize α^k in the k -th iteration (7) satisfies

$$\alpha^k \leq \frac{1}{4B\sqrt{M}} \log \left[1 + \frac{P_1 P_2}{(N_0 + \sum_{i=3}^M P_i)(N_0 + \sum_{i=1}^M P_i)} \right],$$

then at most M constraints of the capacity region C_g can be violated at each iteration step.

Proof: We first show that the inequality in (15) holds for the following choice of δ :

$$\delta = \frac{1}{4} \log \left[1 + \frac{P_1 P_2}{(N_0 + \sum_{i=3}^M P_i)(N_0 + \sum_{i=1}^M P_i)} \right].$$

In order to verify this, rewrite the right hand side of (15) as

$$\begin{aligned} & \frac{1}{4} \log \left[\frac{(N_0 + \sum_{i \in S} P_i)(N_0 + \sum_{i \in T} P_i)}{(N_0 + \sum_{i \in S \cap T} P_i)(N_0 + \sum_{i \in S \cup T} P_i)} \right] \\ &= \frac{1}{4} \log \left[1 + \frac{\sum_{(i,j) \in (S \setminus T) \times (T \setminus S)} P_i P_j}{(N_0 + \sum_{i \in S \cap T} P_i)(N_0 + \sum_{i \in S \cup T} P_i)} \right] \\ &\geq \frac{1}{4} \log \left[1 + \frac{P_1 P_2}{(N_0 + \sum_{i \in S \cap T} P_i)(N_0 + \sum_{i \in S \cup T} P_i)} \right] \\ &\geq \frac{1}{4} \log \left[1 + \frac{P_1 P_2}{(N_0 + \sum_{i \in S \cap T} P_i)(N_0 + \sum_{i=1}^M P_i)} \right] \\ &\geq \frac{1}{4} \log \left[1 + \frac{P_1 P_2}{(N_0 + \sum_{i=3}^M P_i)(N_0 + \sum_{i=1}^M P_i)} \right]. \end{aligned}$$

The inequalities can be justified by using the monotonicity of the logarithm function and the fact that $(S \setminus T) \times (T \setminus S)$ is non-empty because $S \cap T \neq S, T$.

Now, let \mathbf{R}^k be feasible in the capacity region, C_g . For every $S \subseteq \mathcal{M}$, we have

$$\begin{aligned} \sum_{i \in S} (R_i^k + \alpha^k g_i^k) &= \sum_{i \in S} R_i^k + \alpha^k \|g^k\| \sum_{i \in S} \frac{g_i^k}{\|g^k\|} \\ &\leq f(S) + \frac{\delta}{B\sqrt{M}} B \sum_{i \in S} \frac{g_i^k}{\|g^k\|} \\ &\leq f(S) + \delta, \end{aligned} \quad (21)$$

where the first inequality follows from the hypotheses and the second inequality follows from the fact that for any unit vector $d \in \mathbb{R}^M$, it is true that

$$\sum_{i \in S} d_i \leq \sum_{i \in S} |d_i| \leq \sqrt{M}. \quad (22)$$

Thus, if α^k satisfies (21) then $(\mathbf{R}^k + \alpha^k \mathbf{g}^k) \in \mathcal{E}_\delta(C_g)$, for some δ for which (15) holds. Therefore, by Lemma 1 the number of violated constraints does not exceed M . \blacksquare

In view of the fact that a violated constraint can be identified in $O(M^2 \log M)$ time (see the Algorithm in Appendix I), Theorem 2 implies that, for small enough stepsize, the approximate projection can be implemented in $O(M^3 \log M)$ time.

V. CONCLUSION

We addressed the problem of optimal rate allocation in a non-fading multiple access channel from an information theoretic point of view. We formulated the problem as maximizing a general concave utility function of transmission rates over the capacity region of the multiple-access channel.

We presented an iterative gradient projection method for solving this problem. In order to make the projection on a set defined by exponentially many constraints tractable, we considered a method that uses approximate projections. Using the special structure of the capacity region, we showed that the approximate projection can be performed in time polynomial in the number of users.

In ongoing work, we extend our analysis to finding dynamic resource allocation policies in fading multiple-access channels. We study both rate and power allocation policies under different assumptions on the availability of channel statistics information.

APPENDIX I

ALGORITHM FOR FINDING A VIOLATED CONSTRAINT

In this section, we present an alternative algorithm based on rate-splitting idea to identify a violated constraint for an infeasible point. For a feasible point, the algorithm provides information for decoding by successive cancellation. We first introduce some definitions.

Definition 4: The quadruple $(M, \mathbf{P}, \mathbf{R}, N_0)$ is called a *configuration* for an M -user multiple-access channel, where $\mathbf{R} = (R_1, \dots, R_M)$ is the rate tuple, $\mathbf{P} = (P_1, \dots, P_M)$ represents the received power and N_0 is the noise variance. For any given configuration, the *elevation*, $\delta \in \mathbb{R}^M$, is defined as the unique vector satisfying

$$R_i = C(P_i, N_0 + \delta_i), \quad i = 1, \dots, M. \quad (23)$$

Intuitively, we can think of message i as rectangles of height P_i , raised above the noise level by δ_i . In face, δ_i is the amount of additional Gaussian interference that message i can tolerate. Note that if a configuration is feasible then its elevation vector is non-negative, but that is not sufficient to check feasibility.

Definition 5: The configuration $(M, \mathbf{P}, \mathbf{R}, N_0)$ is *single-user codable*, if after possible re-indexing,

$$\delta_{i+1} \geq \delta_i + P_i, \quad i = 0, 1, \dots, M-1, \quad (24)$$

where we have defined $\delta_0 = P_0 = 0$ for convention.

By the graphical representation described earlier, a configuration is single-user codable if the none of the messages are overlapping.

Definition 6: The quadruple $(m, \mathbf{p}, \mathbf{r}, N_0)$ is a *spin-off* of $(M, \mathbf{P}, \mathbf{R}, N_0)$ if there exists a surjective mapping $\phi : \{1, \dots, m\} \rightarrow \{1, \dots, M\}$ such that for all $i \in \{1, \dots, M\}$ we have

$$\begin{aligned} P_i &\geq \sum_{j \in \phi^{-1}(i)} p_j, \\ R_i &\leq \sum_{j \in \phi^{-1}(i)} r_j. \end{aligned}$$

where $\phi^{-1}(i)$ is the set of all $j \in \{1, \dots, m\}$ that map into i by means of ϕ .

Definition 7: A *hyper-user* with power \bar{P} , rate \bar{R} , is obtained by merging d actual users with powers $(P_{i_1}, \dots, P_{i_d})$ and rates $(R_{i_1}, \dots, R_{i_d})$, i.e.,

$$\bar{P} = \sum_{k=1}^d P_{i_k}, \quad \bar{R} = \sum_{k=1}^d R_{i_k}. \quad (25)$$

Proposition 4: For any M -user achievable configuration $(M, \mathbf{P}, \mathbf{R}, N_0)$, there exists a spin-off $(m, \mathbf{p}, \mathbf{r}, N_0)$ which is single user codable.

Proof: See Theorem 1 of [8]. ■

Here, we give a brief sketch of the proof to give intuition about the algorithm. The proof is by induction on M . For a given configuration, if none of the messages are overlapping then the spin-off is trivially equal to the configuration. Otherwise, merge the two overlapping users into a *hyper-user* of rate and power equal the sum rate and sum power of the overlapping users, respectively. Now the problem is reduced to rate splitting for $(M - 1)$ users. This proof suggests a recursive algorithm for rate-splitting that gives the actual spin-off for a given configuration.

It follows directly from the proof of Proposition 4 that this recursive algorithm gives a single-user codable spin-off for an achievable configuration. If the configuration is not achievable, then the algorithm encounters a hyper-user with negative elevation. At this point the algorithm terminates. Suppose that hyper-user has rate \bar{R} and power \bar{P} . Negative elevation is equivalent to the following

$$\bar{R} > C(\bar{P}, N_0).$$

Hence, by Definition 7 we have,

$$\sum_{i \in S} R_i > C\left(\sum_{i \in S} P_i, N_0\right).$$

where $S = \{i_1, \dots, i_d\} \subseteq \mathcal{M}$. Therefore, a hyper-user with negative elevation leads us to a violated constraint in the initial configuration.

The complexity of this algorithm can be computed as follows. The algorithm terminates after at most M recursions. At each recursion, all the elevations are computed in $O(M)$ time and they are sorted in $O(M \log M)$ time. Once the users are sorted by their elevation it takes $O(M)$ time to either find two overlapping users or a hyper-user with negative elevation. Hence, the algorithm runs in $O(M^2 \log M)$ time.

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