Local Indices for Degenerate Variational Inequalities

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We provide an index formula for solutions of variational inequality problems defined by a continuously differentiable function F over a convex set M represented by a finite number of inequality constraints. Our index formula can be applied when the solutions are non-singular and possibly degenerate, as long as they also satisfy the *injective normal map (INM)* property, which is implied by strong stability. We show that, when the INM property holds the degeneracy in a solution can be removed by perturbing the function F slightly, i.e. the index of a degenerate solution is equal to the index of a non-degenerate solution of a slightly perturbed variational inequality problem. We further show that our definition of the index is equivalent to the topological index of the normal map at the zero corresponding to the solution. As an application of our index formula, we provide a global index theorem for variational inequalities which holds even when the solutions are degenerate.

Key words: variational inequality, index theory, complementarity problem, uniqueness

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1. Introduction Let M be a subset of $\mathbb{R}^n$ and $F : M \mapsto \mathbb{R}^n$ be a function. The *variational inequality* problem is to find a vector $x \in M$ such that

$$(y - x)^T F(x) \geq 0, \ \forall \ y \in M. \tag{1}$$

When M is convex, x is a solution to the variational inequality problem iff $-F(x)$ belongs to the normal cone of M at x, which we denote by $N_M(x)$. We say that a solution is *nondegenerate* if $-F(x)$ belongs to the relative interior of $N_M(x)$; and *degenerate* otherwise. The variational inequality problem provides a general framework for the study of optimization and equilibrium problems. Variational inequalities have therefore become an important tool for a range of problems in operations research, economics, finance, and engineering (see Facchinei-Pang [4] or Harker-Pang [8]).

The indices of solutions to the variational inequalities are integer values with useful mathematical properties defined using the axiomatic degree theory or related index theories. Calculating the indices of solutions is important for a number of problems, including the analysis of the uniqueness and stability of solutions.[1] The indices are typically calculated for non-degenerate solutions, yet non-degeneracy is a difficult property to verify. Our main purpose in this paper is to introduce *the injective normal map (INM)* property, which is weaker than existing similar conditions in the literature, and enables calculation of local indices for non-degenerate and degenerate solutions. We then illustrate the use of this approach by deriving a uniqueness result for variational inequalities with possibly degenerate solutions, which generalizes the uniqueness result in our earlier work (cf. Proposition 5.1 in [13]).

In particular, we provide an index formula for the solutions to the variational inequality problem under the conditions that the convex set M is defined by a set of inequalities that satisfy the linear independent constraint qualification, the function F is continuously differentiable, and the solution is non-singular[2] and satisfies the INM property. The INM property essentially requires the normal map to be locally injective. This property is weaker than non-degeneracy, is relatively easy to check, and is implied by the strong stability of the solution. Moreover, there is a class of problems in which the solutions satisfy the INM property but not strong stability [see part (d) of Theorem 4.1 and Example 4.1].[3]

Our main result shows that the index formula we provide for a degenerate solution of a variational inequality problem is equal to the index of a non-degenerate solution to a modified variational inequality problem in which the function F is slightly perturbed in a neighborhood of the solution. In other words, as long as the solution satisfies the INM property, the degeneracy in the solution can be removed by locally perturbing the function F. Using our main result, we also show that our notion of index for a (possibly degenerate) solution to the variational inequality problem is equivalent to the topological index of the normal map at the zero corresponding to the solution, reconciling the concept of index we introduce with the usual notion of index. Existing results then imply that the sum of indices over all solutions is equal to 1, which allows us to establish sufficient conditions for the uniqueness of solutions.

In related works, degree theory has been used to study existence and local stability of solutions to

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[1]A classic reference for axiomatic degree theory is Ortega-Rheinboldt [24]. See Facchinei-Pang [4], Chapters 2 and 5, Cottle-Pang-Stone [3], Chapters 6 and 7 for a treatment of degree theory and its use in analyzing the stability of solutions. See Cottle-Pang-Stone [3], Chapter 6, Kojima-Saigal [9, 10], Gowda [5], Saigal-Simon [14], Kolstad-Mathiesen [11], Simsek-Ozdaglar-Acemoglu [13] for its application on analyzing the solution set and establishing sufficient conditions for global uniqueness of solutions.

[2]Here non-singularity is equivalent to the solution having a non-singular bordered Jacobian. In our earlier paper, [13], we referred to such non-singular solutions as "non-degenerate" and used the term complementary critical point to describe a non-degenerate solution defined above, which is the common usage in the VI literature. Here we make our terminology more congruent with that in the VI literature (compare Definition 2 in [13] with Definition 3.1 in this paper).

[3]Strong stability requires the solutions to change in a Lipschitz continuous way in the exogenous parameters and is often assumed in local sensitivity analysis (see Definition 4.1 and Facchinei-Pang [4], Chapter 5).

degenerate variational inequality problems. Ha [7] used degree theory to establish sufficient conditions for the stability of solutions to the nonlinear complementarity problem. Stewart [15] derived an index formula for the degenerate solutions of a linear complementarity problem, which has been used by Gowda [5] to generalize earlier stability results and to analyze the solution set of a linear complementarity problem. Gowda-Pang [6] provided an index formula for the solutions of a mixed nonlinear complementarity problem, and, using the equivalence of the variational inequality problem with a mixed nonlinear complementarity problem, studied the stability of the solutions to degenerate variational inequality problems. Our notion of the index (cf. Definition 3.3) coincides with the MLCP-degree of a solution defined in [6] when the solution is non-singular and strongly stable, but otherwise applies more generally since the INM property is weaker than strong stability (see Example 4.1).

The rest of the paper is organized as follows. Section 3 defines our index formula for a solution which satisfies the INM property and proves our main result. It also shows the equivalence of our index with the topological index of the normal map. Section 4 establishes the relationships between the INM property and strong solution stability. In particular, it shows that the INM property is implied by strong stability. It also provides a class of problems in which the solution satisfies the INM property but is not necessarily strongly stable. Section 5 applies our index formula and main result to provide a global index theorem for variational inequalities with possibly degenerate solutions.

2. Notation and Preliminaries In this paper, all vectors are viewed as column vectors, and $x^T y$ denotes the inner product of the vectors x and y. We denote the 2-norm as $\|x\| = (x^T x)^{1/2}$. For a given finite set X, we use $|X|$ to denote its cardinality. Given $x \in \mathbb{R}^n$ and $\delta > 0$, $B(x, \delta)$ denotes the open ball with radius δ centered at x. For a given differentiable function f, $\nabla f(x)$ denotes the gradient of f. If f is twice differentiable at x, then $H_f(x)$ denotes the Hessian of f at x. Given $k \leq n$ and an $n \times k$ matrix G with full column rank, we let $\mathcal{V}(G)$ denote the set of all $n \times n - k$ matrices V with full column rank such that $G^T V = 0$ and $V^T V = I$, i.e. the columns of any $V \in \mathcal{V}(G)$ is an orthonormal basis for the null space of G.

We consider a compact region, M, defined by finitely many inequality constraints, i.e.,

$$M = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i \in I = \{1, 2, ..., |I|\}\}, \tag{2}$$

where the $g_i : \mathbb{R}^n \mapsto \mathbb{R}$, $i \in I$, are convex and twice continuously differentiable. For some $x \in M$, let $I(x) = \{i \in I \mid g_i(x) = 0\}$ denote the set of active constraints. We will adopt the following assumption throughout the paper.

Assumption 2.1 The set M is non-empty and every $x \in M$ satisfies the *linear independence constraint qualification*, i.e., for every $x \in M$, the vectors $\{\nabla g_i(x) \mid i \in I(x)\}$ are linearly independent (see Bertsekas-Nedic-Ozdaglar [1], Section 5.4).

We define the *normal space at* x as the subspace of $\mathbb{R}^n$ spanned by the vectors $\{\nabla g_i(x) \mid i \in I(x)\}$. For $x \in M$, we denote the $n \times |I(x)|$ change-of-coordinates matrix from normal coordinates to standard coordinates as

$$G(x) = [\nabla g_i(x)|_{i \in I(x)}],$$

where columns ∇g_i are ordered in increasing order of i. We define the *tangent space at* x as the null space of $G(x)$. Note that the columns of any $V \in \mathcal{V}(G(x))$ constitute a basis for the tangent space.[4]

We next recall the notion of a normal cone which will be used in our analysis (see Clarke [2]):

Definition 2.1 Let M be a region given by (2). Let $x \in M$ with $I(x) \neq \emptyset$. The *normal cone of* M *at* x, $N_M(x)$, is defined by

$$N_M(x) \quad = \quad \{v \in \mathbb{R}^n \mid v = G(x)\lambda,\ \lambda \in \mathbb{R}^{|I(x)|},\ \lambda \geq 0\}.$$

[4]For notational convenience, when the normal space (resp. the tangent space) is zero dimensional, $G(x)$ [resp. $V \in \mathcal{V}(G(x))$] denotes the $n \times 0$ dimensional empty matrix.

We define the *boundary of the normal cone of M at x*, $\mathrm{bd}(N_M(x))$, by

$$\mathrm{bd}(N_M(x)) = N_M(x) - \mathrm{ri}(N_M(x)),$$

where $\mathrm{ri}(N_M(x))$ is the relative interior of the convex set $N_M(x)$, i.e.,

$$\mathrm{ri}(N_M(x)) = \{v \in \mathbb{R}^n \mid v = G(v)\lambda,\ \lambda \in \mathbb{R}^{|I(x)|},\ \lambda > 0\}.$$

If $I(x) = \emptyset$, we define $N_M(x) = \{0\}$ and $\mathrm{bd}(N_M(x)) = \emptyset$.

3. An Index Formula for Degenerate Variational Inequalities We first define the variational inequality problem for the region M, and introduce the notions of non-degeneracy and non-singularity.

Definition 3.1 Let $M \subset \mathbb{R}^n$ be a region given by (2), U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a function.

(a) We say that $x \in M$ is a *solution to the variational inequality problem of F over M* if Eq. (1) holds. Since M is convex, equivalently, x is a solution iff $-F(x) \in N_M(x)$. We denote the set of solutions to the variational inequality problem by $\mathrm{VI}(F, M)$.

(b) For $x \in \mathrm{VI}(F, M)$, we define $\lambda(x) \geq 0$ to be the unique vector in $\mathbb{R}^{|I(x)|}$ that satisfies

$$F(x) + G(x)\lambda(x) = 0. \tag{3}$$

We say that $x \in \mathrm{VI}(F, M)$ is *non-degenerate* if $-F(x) \in \mathrm{ri}(N_M(x))$ and *degenerate* otherwise. In other words, $x \in \mathrm{VI}(F, M)$ is non-degenerate iff $\lambda(x) > 0$.

(c) Let F be continuously differentiable at $x \in \mathrm{VI}(F, M)$. We define $L : U \mapsto \mathbb{R}^n$ with

$$L(x) = F(x) + \sum_{i \in I(x)} \lambda_i(x) \nabla g_i(x) \tag{4}$$

and

$$\Gamma(x, V) = V^T \nabla L(x) V \tag{5}$$

where $V \in \mathcal{V}(G(x))$ is some basis matrix for the tangent space. We say that x is *non-singular* iff $\Gamma(x, V)$ is a non-singular matrix for some $V \in \mathcal{V}(G(x))$.

We next define the Euclidean projection and the normal map corresponding to a variational inequality problem, which we need to define the INM property.

Definition 3.2 Let $M \subset \mathbb{R}^n$ be a closed convex region and $F : M \mapsto \mathbb{R}^n$ be a function.

(a) The projection function $\pi_M : \mathbb{R}^n \mapsto M$ is given by

$$\pi_M(y) = \arg\min_{x \in M} \|y - x\| \tag{6}$$

(b) The normal map associated with $\mathrm{VI}(F, M)$, $F_M^{\mathrm{nor}} : \mathbb{R}^n \mapsto \mathbb{R}^n$, is given by

$$F_M^{\mathrm{nor}}(z) = F(\pi_M(z)) + z - \pi_M(z).$$

Since M is closed and convex, the convex optimization problem in (6) has a unique solution, hence the projection function is well defined. Moreover, the projection function is continuous, and hence F_M^{nor} is also continuous (see parts [a] and [c] of the Projection Theorem in [1]).

We need the following lemma in subsequent analysis, which shows that there is a one-to-one correspondence between solutions of $\mathrm{VI}(F, M)$ and zeros of F_M^{nor} (see Proposition 1.5.9 in Facchinei-Pang [4]).

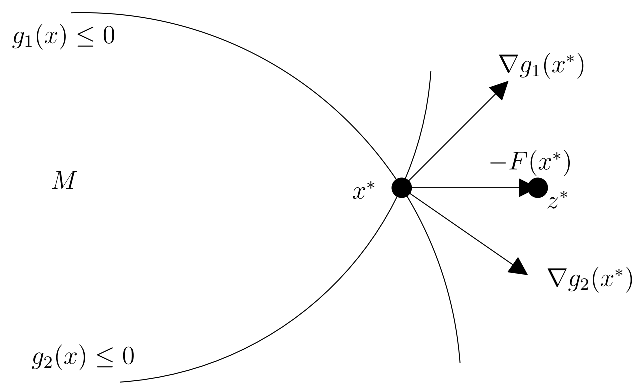

Figure 1: A vector x^* is in $\mathrm{VI}(F,M)$ iff there exists a vector z^* which satisfies $\pi_M(z^*) = x^*$ and $F_M^{\mathrm{nor}}(z^*) = 0$. It can be seen that z^* is unique and is given by $z^* = x^* - F(x^*)$.

Lemma 3.1 Let $M \subset \mathbb{R}^n$ be a closed convex region and $F : M \mapsto \mathbb{R}^n$ be a function. Then, a vector x belongs to $\mathrm{VI}(F,M)$ iff there exists a vector z such that $x = \pi_M(z)$ and $F_M^{\mathrm{nor}}(z) = 0$. In particular, if $x \in \mathrm{VI}(F,M)$, then $z = x - F(x)$ satisfies $\pi_M(z) = x$ and $F_M^{\mathrm{nor}}(z) = 0$ (see Figure 1).

We next introduce the INM property for a solution and define the index for solutions which are non-singular and which satisfy the INM property.

Definition 3.3 Let M be a region given by (2). Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a function. Let $x \in \mathrm{VI}(F,M)$ and $z = x - F(x)$ be the corresponding zero of F_M^{nor} (cf. Lemma 3.1).

(a) The vector x has the *injective normal map* (INM) property iff F_M^{nor} is injective in an open neighborhood of z.

(b) Let x be non-singular with the INM property. We define the index of F at x as

$$\mathrm{ind}_F(x) = \mathrm{sign}(\det(\Gamma(x,V)))$$

for some $V \in \mathcal{V}(G(x))$, where $\Gamma(x,V)$ is defined in Eq. (5).[5]

We show later in Theorem 4.1 that non-degeneracy and non-singularity imply the INM property, hence the index formula in Definition 3.3 generalizes the one we provide for non-degenerate solutions in Simsek-Ozdaglar-Acemoglu [13]. To state our main result, we need the following lemma, which shows that a solution with the INM property is locally unique. The proof follows from Lemma 3.1.

Lemma 3.2 Let M be a region given by (2), U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Assume that $x^* \in \mathrm{VI}(F,M)$ has the INM property. Then, there exists an open set S containing x^* such that $S \cap \mathrm{VI}(F,M) = \{x^*\}$, i.e. x^* is a locally unique solution to the variational inequality problem.

The following theorem is our main result, which shows that the index of a degenerate solution is equal to the index of a non-degenerate solution of a locally perturbed variational inequality, i.e. we can remove the degeneracy in a solution x^* by slightly perturbing the function F in a neighborhood of x^*, provided that x^* has the INM property.

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[5]This definition is independent of the choice of $V \in \mathcal{V}(G(x))$. Let $V_1, V_2 \in \mathcal{V}(G(x))$. Since columns of both V_1 and V_2 are bases for the tangent space, there exist a change of coordinates matrix C_{12} such that $V_1 C_{12} = V_2$. Since $V_1^T V_1 = V_2^T V_2 = I$, we have $C_{12}^T C_{12} = I$, hence C_{12} is orthonormal and $\det(C_{12}) = 1$. Then, $\Gamma(x,V_1) = C_{12}^T \Gamma(x,V_2) C_{12}$ implies $\det(\Gamma(x,V_1)) = \det(\Gamma(x,V_2))$, showing that the index is well defined.

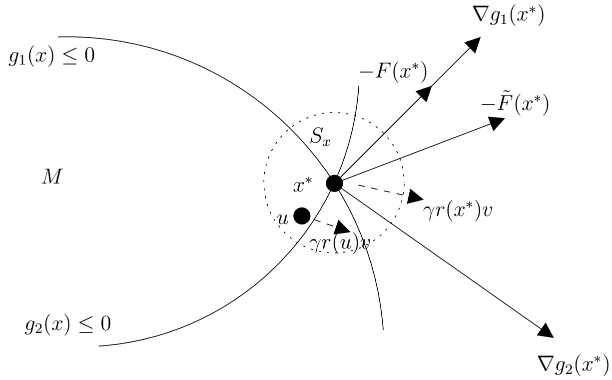

Figure 2: Functions F and $\tilde{F}$ agree everywhere except the open set S_x, over which $\tilde{F}(x) = F(x) - \gamma r(x)v$. Note that x^* is a degenerate solution of $\mathrm{VI}(F, M)$ but a non-degenerate solution of $\mathrm{VI}(\tilde{F}, M)$.

THEOREM 3.1 Let M be a region given by (2), U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Let $x^* \in \mathrm{VI}(F, M)$ be a degenerate solution which has the INM property. Let S be an open set which contains x^* and does not contain any other solution (cf. Lemma 3.2). Then, there exists a function $\tilde{F} : U \mapsto \mathbb{R}^n$ which agrees with F except possibly on S and satisfies the following:

(a) The solution set to the variational inequality problem is unchanged, i.e., $\mathrm{VI}(\tilde{F}, M) = \mathrm{VI}(F, M)$.

(b) The vector x^* is a non-degenerate solution of $\mathrm{VI}(\tilde{F}, M)$. If x^* is a non-singular solution of $\mathrm{VI}(F, M)$, then it is also a non-singular solution of $\mathrm{VI}(\tilde{F}, M)$ and satisfies the INM property; moreover, $\mathrm{ind}_F(x^*) = \mathrm{ind}_{\tilde{F}}(x^*)$.

PROOF. Let $z^* = x^* - F(x^*)$. Since x^* has the INM property, there exists $\epsilon > 0$ small enough such that F_M^{nor} is injective over $B(z^*, 2\epsilon)$. Let $S_x \subset S$ such that

$$x - F(x) \in B(z^*, \epsilon), \qquad \forall\, x \in S_x. \tag{7}$$

In view of Assumption 2.1 and the fact that x^* is degenerate, i.e., $I(x^*) \neq \emptyset$, it follows that the vector

$$\overline{v} = \sum_{i \in I(x^*)} \nabla g_i(x^*)$$

is non-zero. Let $v = \overline{v}/\|\overline{v}\|$. Then, $v \in \mathrm{ri}(N_M(x^*))$. Let $r : U \mapsto \mathbb{R}$ be a continuously differentiable function which satisfies

$$\begin{cases} r(x^*) = 1, \\ r(u) \in [0, 1], \ \text{if } u \in S_x \\ r(u) = 0, \ \text{if } u \notin S_x. \end{cases}$$

Fix some $\gamma \in (0, \epsilon]$ and let $\tilde{F} : U \mapsto \mathbb{R}^n$ be a function given by

$$\tilde{F}(u) = F(u) - \gamma r(u) v,$$

(see Figure 2).

We claim that the function $\tilde{F}$ satisfies the claims of the lemma. First note that $x^* \in \mathrm{VI}(\tilde{F}, M)$, i.e. x^* is also a solution to the perturbed variational inequality. Assume, to get a contradiction, that (a) does not hold. Since $\tilde{F}$ and F differ only on S_x and since $\mathrm{VI}(F, M) \cap S_x = \{x^*\}$, there exists $x' \in \mathrm{VI}(\tilde{F}, M) \cap S_x - \{x^*\}$. Let

$$z' = x' - \tilde{F}(x') = x' - F(x') + \gamma r(x') v. \tag{8}$$

By (7), we have

$$x' - F(x') \in B(z^*, \epsilon). \tag{9}$$

Moreover, by the definition of $r(\cdot)$ and the fact that $\|v\| = 1$, we have

$$\|\gamma r(x')v\| \leq \epsilon, \tag{10}$$

hence $z' \in B(z^*, 2\epsilon)$ by Eqs. (8)-(10) and the triangle inequality. Using Lemma 3.1, we have $\tilde{F}_M^{\text{nor}}(z') = 0$ and $\pi_M(z') = x'$, which yields

$$\begin{aligned} F_M^{\text{nor}}(z') &= \tilde{F}_M^{\text{nor}}(z') + \gamma r(\pi_M(z'))v \\ &= \gamma r(x')v. \end{aligned} \tag{11}$$

Let $\bar{z} = z^* + \gamma r(x')v$. Since $\|\gamma r(x')v\| \leq \epsilon$, we have $\bar{z} \in B(z^*, 2\epsilon)$. Moreover, since $\gamma r(x')v \in N_M(x^*)$ and $z^* - x^* = -F(x^*) \in N_M(x^*)$, we have

$$\bar{z} - x^* = z^* - x^* + \gamma r(x')v \in N_M(x^*)$$

and hence $\pi_M(\bar{z}) = x^*$. Then,

$$\begin{aligned} F_M^{\text{nor}}(\bar{z}) &= F(\pi_M(\bar{z})) + \bar{z} - \pi_M(\bar{z}) \\ &= F(x^*) + z^* + \gamma r(x')v - x^* \\ &= \gamma r(x')v, \end{aligned} \tag{12}$$

where the last equality follows since $z^* = x^* - F(x^*)$.

Since $z', \bar{z} \in B(z^*, 2\epsilon)$ and F_M^{nor} is injective over $B(z^*, 2\epsilon)$, Eqs. (11) and (12) imply that $z' = \bar{z}$. But then,

$$x' = \pi_M(z') = \pi_M(\bar{z}) = x^*,$$

yielding a contradiction, proving part (a).

To prove part (b), note that $-F(x^*) \in N_M(x^*)$ and $\gamma r(x^*)v \in \text{ri}(N_M(x^*))$. Then we have

$$-\tilde{F}(x^*) = \gamma r(x^*)v - F(x^*) \in \text{ri}(N_M(x^*)),$$

hence, x^* is a non-degenerate solution in $\text{VI}(\tilde{F}, M)$. We further have, $V^T v = 0$ since $v \in N_M(x^*)$. Then, using $\nabla \tilde{F}(x^*) = \nabla F(x^*) + \gamma v \nabla r(x^*)^T$, we have

$$\begin{aligned} \Gamma_{\tilde{F}}(x^*, V) &= V^T \left(\nabla \tilde{F}(x^*) + \sum_{i \in I(x^*)} \lambda_i(x^*) H_{g_i}(x^*) \right) V \\ &= V^T \left(\nabla F(x^*) + \sum_{i \in I(x^*)} \lambda_i(x^*) H_{g_i}(x^*) \right) V + \gamma V^T v \nabla r(x^*)^T V \\ &= \Gamma_F(x^*, V). \end{aligned}$$

Hence, if x^* is non-singular as a solution of $\text{VI}(F, M)$, then it is non-singular as a solution of $\text{VI}(\tilde{F}, M)$ and thus also satisfies the INM property by Theorem 4.1. Moreover,

$$\text{ind}_{\tilde{F}}(x^*) \quad = \quad \text{sign}(\det(\Gamma_{\tilde{F}}(x^*))) = \text{sign}(\det(\Gamma_F(x^*))) = \text{ind}_F(x^*)$$

completing the proof of part (b). **Q.E.D.**

We next show that our notion of the index of a solution of a variational inequality problem is equivalent to the topological index of the normal map at the corresponding zero. For a given set S, let $\text{bd}(S)$ denote the boundary of S and $\text{cl}(S)$ denote the closure of S. Let $S \subset \mathbb{R}^n$ be a bounded open set and $f : \text{cl}(S) \mapsto \mathbb{R}^n$ be a continuous function. For a vector $p \notin f(\text{bd}(S))$, *the topological degree of f at p relative to S* is an axiomatically defined integer, which we denote by $\deg(f, S, p)$ (see Section 2.1 of Facchinei-Pang [4] or Chapter 6 of Ortega-Rheinboldt [24]). The degree provides information regarding the solutions of the equation $f(x) = p$ over the set S. Consider a vector $x \in S$ that is an isolated solution to the equation $f(x) = 0$, which we call an isolated zero of f. It is well known that for all sufficiently small neighborhoods S_x of x, $\deg(f, S_x, 0)$ is the same. This common degree is called *the topological index of f at x*, which we denote by $\mathrm{index}(f,x)$. The following theorem establishes that our definition of the index of a solution to the variational inequality problem is equal to the topological index of the normal map at the corresponding zero.[6]

THEOREM 3.2 Let M be a region given by (2), U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Let $x^* \in \mathrm{VI}(F, M)$ be a solution which is non-singular and has the INM property. Let $z^* = x^* - F(x^*)$. Then,

$$\mathrm{ind}_F(x^*) = \mathrm{index}(F_M^{\mathrm{nor}}, z^*).$$

PROOF. Let $\epsilon > 0$ be the constant and S_x be the neighborhood of x^* defined in the proof of Theorem 3.1. Let $\epsilon' \in (0, \epsilon)$ be sufficiently small that $\pi(z) \in S_x$ for all $z \in \mathrm{cl}(B(z^*, \epsilon'))$. Let $S_z = B(z^*, \epsilon')$. We index the perturbed function used in the proof by γ, i.e. $\tilde{F}_\gamma(u) = F(u) - \gamma r(u)v$ for $\gamma \in [0, \epsilon]$ and we define $\tilde{F} = \tilde{F}_{\epsilon'/2}$. Let $z_\gamma^* = z^* + \gamma r(x^*)v$ and $\tilde{z}^* = z_{\epsilon'/2}^*$. Note that $\tilde{F}_0 = F$ and $z_0^* = z^*$.

We claim that, for $\gamma \in [0, \epsilon'/2]$, z_γ^* is the only zero of $(\tilde{F}_\gamma)_M^{\mathrm{nor}}$ over $\mathrm{cl}(S_z)$. The vector z_γ^* is a zero of $(\tilde{F}_\gamma)_M^{\mathrm{nor}}$ by construction. Suppose $(\tilde{F}_\gamma)_M^{\mathrm{nor}}(z') = 0$ where $z' \in \mathrm{cl}(S_z)$. Since $\pi(z') \in S_x$ and since, by the proof of Theorem 3.1, x^* is the only solution of $\mathrm{VI}(\tilde{F}_\gamma, M)$ over S_x, it must be the case that $\pi(z') = x^*$. But then, $z' = x^* - \tilde{F}_\gamma(x^*) = z_\gamma^*$, hence z_γ^* is the only zero over $\mathrm{cl}(S_z)$. Then,

$$\mathrm{index}\left((\tilde{F}_\gamma)_M^{\mathrm{nor}}, z_\gamma^*\right) = \deg\left((\tilde{F}_\gamma)_M^{\mathrm{nor}}, S_z, 0\right), \tag{13}$$

by definition of the topological index. Since for any $\gamma \in [0, \epsilon'/2]$, $(\tilde{F}_\gamma)_M^{\mathrm{nor}}$ does not have a zero on the boundary of $\mathrm{cl}(S_z)$ and since $(\tilde{F}_\gamma)_M^{\mathrm{nor}}$ defines a homotopy between F_M^{nor} and $\tilde{F}_M^{\mathrm{nor}}$, it follows by a standard property of the topological degree (cf. Proposition 2.1.3 in Facchinei-Pang [4]) that

$$\deg(F_M^{\mathrm{nor}}, S_z, 0) = \deg(\tilde{F}_M^{\mathrm{nor}}, S_z, 0),$$

which, by Eq. (13) implies that

$$\mathrm{index}(F_M^{\mathrm{nor}}, z^*) = \mathrm{index}(\tilde{F}_M^{\mathrm{nor}}, \tilde{z}^*). \tag{14}$$

By Theorem 3.1, x^* is a non-degenerate solution of $\tilde{F}$. Then, Theorem 4.1 in Simsek-Ozdaglar-Acemoglu [13] implies that $\tilde{F}_M^{\mathrm{nor}}$ is continuously differentiable at $\tilde{z}^*$ and the sign of the determinant of the Jacobian satisfies, in the notation of this paper,

$$\mathrm{sign}(\det(\nabla \tilde{F}_M^{\mathrm{nor}}(\tilde{z}^*))) = \mathrm{sign}(\det(\Gamma_{\tilde{F}}(x^*, V))) = \mathrm{ind}_{\tilde{F}}(x^*) = \mathrm{ind}_F(x^*), \tag{15}$$

where the last equality follows by Theorem 3.1. Moreover, since $\tilde{F}_M^{\mathrm{nor}}$ is continuously differentiable at z^*, the topological index satisfies

$$\mathrm{index}(\tilde{F}_M^{\mathrm{nor}}, \tilde{z}^*) = \mathrm{sign}(\det(\nabla \tilde{F}_M^{\mathrm{nor}}(\tilde{z}^*))). \tag{16}$$

The result follows by Eqs. (14)-(16). **Q.E.D.**

4. The INM Property and the Strong Solution Stability for the Variational Inequality Problem In this section, we investigate the relationship of the INM property with similar conditions studied in the literature, in particular, strong stability. For a given function $F : U \mapsto \mathbb{R}^n$, a given set S, and a scalar ϵ, let $\mathbb{B}(F, \epsilon, S)$ denote the set of continuous functions $G : U \mapsto \mathbb{R}^n$ such that

$$\sup_{y \in S} \|G(y) - F(y)\| \leq \epsilon.$$

The stability of the solution to a variational inequality is defined as follows (cf. Facchinei-Pang [4], Definition 5.3.1).

[6] We thank an anonymous referee for pointing out this result.

Definition 4.1 Let M be a region given by (2), U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a function. Let x be a solution vector in $\mathrm{VI}(F, M)$. The solution x is *strongly stable* if for every open neighborhood $\mathcal{N}$ of x such that $\mathcal{N} \subseteq U$ and $\mathrm{VI}(F, M) \cap \mathcal{N} = \{x\}$, there exists two positive scalars c and ϵ such that, for any two functions $G, H \in \mathbb{B}(F, \epsilon, M \cap \mathrm{cl}\mathcal{N})$,

$$\mathrm{VI}(G, M) \cap \mathcal{N} \neq \emptyset, \ \mathrm{VI}(H, M) \cap \mathcal{N} \neq \emptyset \tag{17}$$

and for every $x' \in \mathrm{VI}(G, M) \cap \mathcal{N}$ and $x'' \in \mathrm{VI}(H, M) \cap \mathcal{N}$,

$$\|x' - x''\| \leq c \, \|e_G(x') - e_H(x'')\| \,, \tag{18}$$

where $e_G(x) = F(x) - G(x)$ and $e_H(x) = F(x) - H(x)$ respectively denote the error functions for G and H. In other words, a solution is strongly stable if there exists nearby solutions for slightly perturbed variational inequalities [cf. Eq. (17)] and the nearby solutions satisfy a Lipschitzian property with respect to the amount of perturbation [cf. Eq. (18)].

The following theorem provides conditions which imply the INM property. In particular, parts (a) and (b) show that strongly stable solutions satisfy the INM property and characterize strong stability with a condition on the Jacobian of F which is easier to check. Part (c) shows that non-degeneracy and non-singularity imply strong stability (and hence the INM property), thus our index notion in this paper generalizes the definition of index in Simsek-Ozdaglar-Acemoglu [13]. Part (d) provides a set of conditions which imply the INM property but not necessarily strong stability, demonstrating that the INM property holds in a class of problems in which solutions are not strongly stable.

We need the following notation to state the theorem. For some $x \in M$, let the index sets $\alpha(x)$ and $\beta(x)$ be given by

$$\alpha(x) = \{i \in I \mid \lambda_i(x) > 0 = g_i(x)\}, \tag{19}$$

$$\beta(x) = \{i \in I \mid \lambda_i(x) = 0 = g_i(x)\}, \tag{20}$$

where $\lambda(x)$ is defined in Eq. (3). Let $\mathcal{B}(x)$ be the set of matrices defined by

$$\mathcal{B}(x) = \Big\{ B \mid B = [\nabla g_i(x)]_{i \in J}, \ \alpha(x) \subseteq J \subseteq \alpha(x) \cup \beta(x) \Big\}.$$

Theorem 4.1 Let M be a region given by (2), U be an open set containing M, and $F : U \mapsto \mathbb{R}^n$ be a continuous function. Let $x^* \in \mathrm{VI}(F, M)$ and F be continuously differentiable at x^*.

(a) If x^* is strongly stable, then it satisfies the INM property.

(b) The solution x^* is strongly stable iff all matrices of the form

$$V_B^T \nabla L(x) V_B \tag{21}$$

have the same non-zero determinantal sign for $B \in \mathcal{B}(x)$ and $V_B \in \mathcal{V}(B)$, where $L(x)$ is defined in Eq. (4).

(c) If x^* is non-degenerate and non-singular, then it is strongly stable.

(d) Let S be a neighborhood of x^* such that F is continuously differentiable at every $x \in S \cap M$ and $\nabla F(x)$ is positive definite for every $x \in S \cap M - \{x^*\}$. Then x^* satisfies the INM property.

Proof. Let x^* be a strongly stable solution and let $z^* = x^* - F(x^*)$. By Theorem 5.3.24 in Facchinei-Pang [4], F_M^{nor} is a locally Lipschitz homeomorphism at z^*. In particular, it is locally injective and x^* satisfies the INM property, proving part (a).

To prove part (b), we will show that, given $B \in \mathcal{B}(x)$ and $V_B \in \mathcal{V}(B)$, the matrix $V_B^T \nabla L(x) V_B$ has the same determinantal sign as

$$\nabla L(x) \left(I - B(B^T B)^{-1} B^T \right) + B(B^T B)^{-1} B^T. \tag{22}$$

The result then follows from Theorem 5.3.24 and Proposition 4.2.7 in [4]. Let $B(B^TB)^{-1}B^T = P_B$ for ease of notation. Note that $[V_B\ B]$ is a non-singular $n \times n$ matrix by choice of V_B. Then, the matrix in (22) has the same determinantal sign as

$$[V_B\ B]^T\,[\nabla L(x)(I - P_B) + P_B]\,[V_B\ B].$$

Since $V_B^TB = 0$, we have $V_B^TP_B = 0$ and $P_BV_B = 0$. Using these equations and $B^T(I - P_B) = 0$, the previous expression simplifies to

$$\begin{bmatrix} V_B^T\nabla L(x)V_B & 0 \\ S & B^TB \end{bmatrix}$$

for some matrix S. Since $B \in \mathcal{B}(x)$, B has full rank and $\det(B^TB) > 0$. Therefore, the sign of the determinant of the previous matrix is equal to the sign of the determinant of $V_B^T\nabla L(x)V_B$, completing the proof of part (b).

To prove part (c), note that $\beta(x^*) = \emptyset$ since x^* is non-degenerate. Hence, $\mathcal{B}(x^*)$ has a unique element, i.e. $\mathcal{B}(x^*) = \{G(x^*)\}$. Since x^* is non-singular, $V^T\nabla L(x^*)V$ has the same non-zero determinantal sign for all $V \in \mathcal{V}(G(x^*))$ (which is equal to $\text{ind}_F(x^*)$). The result follows from part (b).

To prove part (d), let $S' \subset S$ be a convex neighborhood of x^*. Let $z^* = x^* - F(x^*)$ and let S_z be a neighborhood of z^* such that $\pi(z) \in S'$ for all $z \in S_z$. Suppose x^* does not have the INM property. Then, there exists $z_1, z_2 \in S_z$ such that $z_1 \neq z_2$ and

$$F_M^{\text{nor}}(z_1) = F_M^{\text{nor}}(z_2). \tag{23}$$

Since $F_M^{\text{nor}}(z) = F(\pi(z)) + z - \pi(z)$, Eq. (23) implies $\pi(z_1) \neq \pi(z_2)$. Let $v = \pi(z_2) - \pi(z_1)$. Note that by convexity of the region M, v (resp. $-v$) is an interior direction at $\pi(z_1)$ (resp. $\pi(z_2)$), hence

$$(z_1 - \pi(z_1))^Tv \leq 0, \text{ and } (z_2 - \pi(z_2))^Tv \geq 0. \tag{24}$$

For $t \in [0,1]$, let $l(t) = \pi(z_1) + tv$ parameterize the line segment connecting $\pi(z_1)$ and $\pi(z_2)$. Consider the real valued function $F_v(x) = F(x)^Tv$. Since $S' \cap M$ is convex, $l(t)$ lies in $S' \cap M \subset S \cap M$ for all $t \in [0,1]$, hence $F_v(l(t))$ is continuously differentiable at each $t \in [0,1]$, which implies

$$\begin{aligned} F(\pi(z_2))^Tv = F_v(\pi(z_2)) &= F(\pi(z_1))^Tv + \int_0^1 v^T\nabla F(l(t))^Tv\,dt \\ &> F(\pi(z_1))^Tv \\ &= F(\pi(z_2))^Tv + (z_2 - \pi(z_2))^Tv - (z_1 - \pi(z_1))^Tv \\ &\geq F(\pi(z_2))^Tv, \end{aligned}$$

where the first inequality follows from the fact that $\nabla F(l(t))$ is positive definite for every $t \in [0,1]$ (except possibly one point if $x^* = l(t')$ for some t'), the last equality follows by Eq. (23) and the last inequality follows from Eq. (24). This yields a contradiction, hence x^* satisfies the INM property, proving part (d). **Q.E.D.**

The following example illustrates that a solution x^* which satisfies the assumptions of part (d) of Theorem 4.1 is not necessarily strongly stable. It also demonstrates that the INM property can be useful in establishing the index of a solution that is degenerate and not strongly stable.

Example 4.1 Let M be a region given by (2) and U be an open set containing M. Let $x^* \in M$ be such that $I(x^*) \neq 0$, i.e. at least one constraint is binding at x^*. Let $E : U \mapsto \mathbb{R}^n$ be a continuously differentiable function such that $E(x^*) = 0$ and $\nabla E(x^*)$ is positive definite. Consider the function $F : U \mapsto \mathbb{R}^n$ given by

$$F(x) = \|E(x)\|^2E(x).$$

Then, $x^* \in \text{VI}(F, M)$ is a degenerate solution. Moreover, there exists a sufficiently small neighborhood S of x^* such that E is a homeomorphism on S and hence $E(x) \neq 0$ for $x \in S - \{x^*\}$. Then, $\nabla F(x) = 2E(x)E(x)^T + \|E(x)\|^2\nabla E(x)$ is positive definite for $x \in S - \{x^*\}$. Hence, F satisfies the assumptions of part (d) of Theorem 4.1 and x^* has the INM property. Since $F(x^*) = 0$ and M satisfies the LICQ property, we have $\lambda(x^*) = 0$ (cf. Definition 3.1), which implies that $\alpha(x^*) = \emptyset$ [cf. Eq. (19)] and that the empty matrix [ ] is in $\mathcal{B}(x^*)$. Moreover, $\nabla L(x^*) = \nabla F(x^*) = 0$, hence Eq. (21) fails for $B = [\,]$ and $V_B = I$. Thus, x^* is not strongly stable. Finally, if $I(x^*) = n$, i.e. the tangent space at x^* is zero dimensional, then x^* is non-singular and has index 1, i.e. our index formula applies despite the fact that x^* is degenerate and not strongly stable.

5. A Global Index Theorem for Variational Inequalities In this section, we provide a global index theorem which is a generalization of Theorem 3.1 in [13] to variational inequalities with possibly degenerate solutions.

Theorem 5.1 Let M be a region given by (2). Let U be an open set containing M and $F : U \mapsto \mathbb{R}^n$ be a continuous function which is continuously differentiable at every $x \in \mathrm{VI}(F, M)$. If every $x \in \mathrm{VI}(F, M)$ is non-singular and has the INM property, then $\mathrm{VI}(F, M)$ has a finite number of elements. Moreover:

$$\sum_{x \in \mathrm{VI}(F,M)} \mathrm{ind}_F(x) = 1. \tag{25}$$

Proof of Theorem 5.1. Using continuity of F and π, it can be seen that $\mathrm{VI}(F, M)$ is a compact set. This further implies that $\mathrm{VI}(F, M)$ has a finite number of elements since each $x \in \mathrm{VI}(F, M)$ is an isolated solution by Lemma 3.2. Let $\mathcal{D}(F, M) \subset \mathrm{VI}(F, M)$ denote the set of degenerate solutions. Let $F^0 = F$ and, for any $j \geq 0$ such that $\mathcal{D}(F^j, M) \neq \emptyset$, define $F^{j+1} = \tilde{F^j}$ where $\tilde{F^j}$ is the perturbed function which satisfies the claims of Theorem 3.1 for the function F^j and an arbitrary $x^* \in \mathcal{D}(F^j, M)$. By Theorem 3.1,

$$|\mathcal{D}(F^{j+1}, M)| = |\mathcal{D}(F^j, M)| - 1.$$

Since $\mathrm{VI}(F, M)$ has finitely many elements, $\mathcal{D}(F, M)$ has finitely many elements, which implies that there exists an integer $m \geq 0$ such that $\mathcal{D}(F^m, M) = \emptyset$. We let $G = F^m$. Then every vector x in $\mathrm{VI}(G, M)$ is non-degenerate and non-singular, moreover, $\mathrm{ind}_G(x) = \mathrm{ind}_F(x)$. Hence Theorem 3.1 in [13] applies to (G, M) and we have $\sum_{x \in \mathrm{VI}(G,M)} \mathrm{ind}_G(x) = 1$, which further implies Eq. (25). **Q.E.D.**

Remark 5.1 Combining Theorem 5.1 and part (b) of Theorem 4.1 provides conditions on the Jacobian of F which are easy to check and which guarantee uniqueness of solutions to the variational inequality problem. In particular, assume that for every $x \in \mathrm{VI}(F, M)$,

$$\det(V_B^T \nabla L(x) V_B) > 0, \tag{26}$$

for all $B \in \mathcal{B}(x)$ and $V_B \in \mathcal{V}(B)$. Then, every $x \in \mathrm{VI}(F, M)$ is non-singular, satisfies the INM property, and has index equal to 1. Hence Theorem 5.1 implies that $\mathrm{VI}(F, M)$ has a unique solution. For the (bounded) mixed complementarity problem, which is the variational inequality problem in which the region M is a bounded rectangle, the sufficient condition in Eq. (26) is equivalent to checking that certain principal minors of the Jacobian of F are positive at each solution. For the nonlinear complementarity problem which is the variational inequality problem in which the region M is the non-negative orthant, an appropriate boundary condition (which allows us to cast the problem as a bounded mixed complementarity problem) and the condition that certain principal minors of the Jacobian F are positive at each solution imply uniqueness. For the linear complementarity problem, which is a special case of the nonlinear complementarity problem in which $F(x) = Ax$ for some matrix A, a sufficient condition for uniqueness is that the matrix A has positive principal minors (i.e. A is a P-matrix).

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