

Discrete Opinion Dynamics with Stubborn Agents*

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We study discrete opinion dynamics in a social network with "stubborn agents" who influence others but do not change their opinions. We generalize the classical voter model by introducing nodes (stubborn agents) that have a fixed state. We show that the presence of stubborn agents with opposing opinions precludes convergence to consensus; instead, opinions converge in distribution with disagreement and fluctuations. In addition to the first moment of this distribution typically studied in the literature, we study the behavior of the second moment in terms of network properties and the opinions and locations of stubborn agents. We also study the problem of "optimal placement of stubborn agents" where the location of a fixed number of stubborn agents is chosen to have the maximum impact on the long-run expected opinions of agents.

Key words: opinion dynamics in networks; the voter model with stubborn agents; social networks

1. Introduction

The most common approach to opinion formation, which either build on Bayesian approaches or non-Bayesian consensus-type ideas, predict that opinion dynamics should converge to agreement (consensus). In practice, we do not see such agreement. Instead, most societies appear to exhibit persistent disagreement, even on topics on which there is frequent communication (see Acemoglu

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and Ozdaglar (2010) for a recent survey).

In this paper, we investigate a model of discrete (in particular binary) opinion dynamics based on the well-known *voter model* originally developed independently by Holley and Liggett (1975) and Clifford and Sudbury (1973). In this model, each individual holds one of two opinions (for example, pro- or anti-government). Society is modeled as a social network (represented by a graph) in which each individual has a set of fixed neighbors with whom he or she may communicate. At each date, some randomly chosen individual observes or communicates one of his or her neighbors and adopts his opinion. In addition to capturing a simple form of opinion dynamics, the voter model is useful as a general framework for the analysis of diffusion of practices, innovations, consumption decisions and epidemics.

In this paper, we generalize the voter model by introducing *stubborn agents*, who have fixed opinions and thus do not update (in terms of other possible applications of the voter model, the stubborn agents would correspond to non-susceptible subsets that still influence the behavior of others). We characterize the effect of network structure and the opinion of stubborn agents (and their network position) on the long-run distribution of opinions. We also discuss how to optimally influence opinion dynamics in such a setting.

In this model, stubborn agents with opposing views prevent consensus in society. Moreover, the opinion of each individual does not necessarily settle into a fixed value. Therefore, our generalized voter model generates the kind of persistent disagreement that seems to be a better description of opinion dynamics in practice. Our first result shows that despite fluctuations, agent opinions converge in distribution. Notably, the long-run (limiting stationary) distribution of opinions depends only on network structure and the opinions and locations of stubborn agents; the initial opinions of non-stubborn (regular) agents have no impact on long-run distribution of opinions. This is intuitive in view of the fact that non-stubborn agents do respond to the opinions of their neighbors while stubborn agents do not.

Our second result characterizes the expected opinion of each agent in the society in the long run, given by the first moment of the stationary distribution of agent opinions, in terms of the

properties of random walks on the underlying network. We show that the expected opinion of each agent can be described in terms of the absorption probabilities of a random walk where stubborn individuals correspond to absorbing states.

Our third result characterizes the fluctuations in agent opinions in the long run in terms of the properties of *coalescing* random walks on the underlying network. We use the variance of the average opinion of the society as a measure of fluctuations. We provide an explicit characterization of the variance in terms of the solution of harmonic equations defined on a product graph, which captures the dynamics of coalescing random walks on the social network. Using this characterization, we then provide estimates (upper bounds) on the variance as a function of the network structure as well as the locations of the stubborn agents. Our estimates show a tight link between the opinion dynamics of individuals that are highly *influential* (central) and the overall extent of fluctuations in society. This also enables us to show how the details of the network structure can turn a society with a relatively stable distribution of opinions (relatively limited fluctuations) to one with unstable opinions (high fluctuations).

Finally, we use this framework to study the problem of optimal stubborn agent placement. Imagine that we are able to choose the location of k stubborn agents of one type in an otherwise known network and we would like to do this to achieve maximum impact on the limiting expected opinions. We show that this optimization problem has a solution in closed form in terms of the fundamental matrix of a Markov process for $k = 1$. For $k > 1$, we introduce a greedy algorithm for the maximization problem and bound the level of suboptimality of the resulting solution.

Our paper is related to a large and growing literature studying diffusion of innovations, opinions and epidemics on networks. The seminal work by Bass (1969) provides a tractable mathematical framework for the analysis of such diffusion processes in the absence of underlying network interactions. The diffusion of diseases are more commonly studied by Bailey (1975), Kermack and McKendrick (1991) under *susceptible, infected, susceptible* (SIS) and *susceptible, infected, removed* (SIR) models, respectively. In both SIS and SIR models, the diffusion is explained through the process of already-infected agents infecting their susceptible neighbors, which is generally assumed

to be stochastic. Kleinberg *et al.* consider related models of decision of new products over networks (Kempe et al. (2003, 2005), Kleinberg (2007)).

Diffusion and the dynamics of opinions in social networks have been studied in the related framework introduced by DeGroot (1974), where opinions of individuals are modeled as continuous variables (which may correspond to beliefs about certain underlying state variable or the probability that a given statement is true). The interaction patterns are captured through near neighbor based linear updates. Because the updates of the DeGroot model has the exact same form as the synchronous average consensus algorithms (see Tsitsiklis (1984)), its analysis is relatively straightforward (see, for example, Golub and Jackson (2010)). Acemoglu *et al.* consider an extension of the DeGroot model where updates are asynchronous and certain individuals are spreading misinformation by not updating their own beliefs (Acemoglu et al. (2010c)). Under the assumption that *no man is an island*, *i.e.*, each and every individual will update his belief with a non-zero probability, they show that society will converge to a consensus.

Finally, another literature also discusses opinion dynamics in a Bayesian setting, where individuals either observe the actions of others and/or communicate with them and update their beliefs about an underlying state variable (see, for example, Acemoglu et al. (2010a), Banerjee (1992), Jackson (2008)).

The binary voter algorithm, which our proposed model is based on, was introduced independently in Holley and Liggett (1975) and Clifford and Sudbury (1973). In this model, each agent in the society can have two possible positions on a given subject, $\{0\}$ or $\{1\}$. Periodically, each individual reassesses his opinion in the following way: He chooses one of his neighbors randomly and adopts the chosen neighbors behavior. There exist a vast volume of literature on the limiting behavior of the binary voter model on both infinite lattices and finite heterogenous topologies (Cox and Griffeath (1986), Gray (1986), Krapivsky (1992), Liggett (2005), Sood and Redner (2005)). However, the exact model we are interested in contains two sets of *stubborn* nodes which hold distinct views and do not change their opinions at all. These individuals can be thought of islands who continuously influence their neighbors through the original voter model update rule, but never

update their decisions. This particular idea was first proposed by Mobilia et al. (2007), Mobilia (2003), and the behavior of the system has been analyzed on certain homogenous graphs by mean field approximation. Wu *et al.* developed an analysis based on averaging over random graphs as well as mean field approximation to take into account the structure of the underlying social network (Wu and Huberman (2004)), while Chinellato *et al.* also use mean field approximation based analysis for a more general class of models (Chinellato et al. (2007)). However, given the dependence of opinion dynamics on the location of stubborn agents, mean field approximation and/or averaging over random topologies are less likely to be reliable in this context.

Finally, Acemoglu *et al.* study opinion fluctuations and disagreement over a general social network in a model with continuous opinions (Acemoglu et al. (2010b)). Their results are complementary to those presented here and use a different mathematical approach. Our model focuses on dynamics of discrete opinions and provides a general characterization of the relationship between the location of stubborn agents and the long-run opinion distribution. The analysis of optimal placement of stubborn agents is also considered only in this paper.

The rest of the paper is organized as follows. Basic notation and terminology are discussed in Section 1.1. In Sections 2.1 and 2.2, we introduce our model and show that agent opinions converge in distribution. We review the dual approach for the original voter model and introduce a new dual for the voter model with stubborn agents in Section 2.3. In Section 3, we focus on the average opinion of the society and show that the average opinion converges in distribution. In Sections 3.1 and 3.2, we characterize the first two moments of the limiting distribution of the average opinion and introduce corresponding bounds. We investigate the behavior of our model in two special graphs in Section 4. We introduce a convergence time measure for our model in Section 5. In Section 6, we focus on the optimal agent placement problem and discuss our results in detail. Finally, we conclude in Section 7. To ease exposition of our results, we decided to relegate the proofs to the Appendix.

1.1. Notation and Terminology

A vector will be denoted by a lower case letter, and all vectors are assumed to be column vectors unless stated otherwise. For a given vector x , x_i denotes the i -th element of the vector x . x' denotes the transpose of x . Matrices will be denoted by upper case letters, and for a given matrix T , $[T]_{ij}$ denotes the entry in i -th row and j -th column. T^k will denote the k -th power of a given matrix T . $\mathbb{P}(\cdot)$ and $\mathbb{E}\{\cdot\}$ denote the probability and the expectation of their arguments, respectively. Equality relations among random variables will denote equivalence in probability measures unless stated otherwise. e represents the exponential constant and $\log(\cdot)$ denotes the natural logarithm. Denoting N as the size of the society, and $f(N), g(N) : \mathbb{N} \rightarrow \mathbb{R}^+ \cup \{0\}$, asymptotic bounds are defined as:

$$f(N) \in O(g(N)) \Rightarrow \text{there exists } k, N_0 > 0, \text{ such that for all } N > N_0, f(N) \leq g(N)k,$$

$$f(N) \in \omega(g(N)) \Rightarrow \text{for all } k, \text{ there exists } N_0 > 0, \text{ such that for all } N > N_0, f(N) \geq g(N)k,$$

$$f(N) \in \Theta(g(N)) \Rightarrow \text{there exists } k_1, k_2, N_0 > 0, \text{ such that for all } N > N_0, g(N)k_1 \leq f(N) \leq g(N)k_2.$$

2. The Voter Model and the Dual Approach

2.1. The Binary Voter Model with Stubborn Agents

We consider a set of agents $\mathcal{V} = \{1, \dots, N\}$ situated in a social network represented by a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where \mathcal{E} is the set of edges representing the connectivity among these individuals.¹

An edge $(i, j) \in \mathcal{E}$ is considered to be directed from i to j ; j is called the head and i is called the tail of the edge. We define the neighbor set of agent $i \in \mathcal{V}$ as $\mathcal{N}_i = \{j \mid (i, j) \in \mathcal{E}\}$. We assume that there are two nonempty disjoint sets of *stubborn agents* $\mathcal{V}_0, \mathcal{V}_1 \subset \mathcal{V}$. Stubborn agents never change their opinions while the non-stubborn agents update their opinions based on the information they receive from their neighbors.

We use $x_i(t)$ to denote the opinion (state) of agent i at time t and $x(t) = [x_i(t)]_{i=1, \dots, N}$ to denote the agent opinion (state) vector at time t . We assume that each non-stubborn agent i starts from an arbitrary initial value $x_i(0) \in \{0, 1\}$ and updates his opinion according to the *binary voter model*

(see Holley and Liggett (1975)).² Each non-stubborn agent $i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$ reassesses his opinion according to a rate $\gamma = 1$ Poisson process independently. At each Poisson arrival, he chooses one of his neighbors uniformly at random and adopts his decision. Each stubborn agent is also recognized according to a rate $\gamma = 1$ Poisson process independently, however he never changes his opinion, *i.e.*,

$$x_i(t) = \begin{cases} 0 & \text{for all } t \geq 0, \text{ if } i \in \mathcal{V}_0 \\ 1 & \text{for all } t \geq 0, \text{ if } i \in \mathcal{V}_1. \end{cases} \quad (1)$$

We refer to this model of opinion dynamics as the *(binary) voter model with stubborn nodes*.

Our goal in this paper is to characterize the asymptotic behavior of the agent opinions $x(t)$ in terms of the underlying graph \mathcal{G} and the location of stubborn agent sets \mathcal{V}_1 and \mathcal{V}_0 .

We adopt the following assumption throughout the paper:

ASSUMPTION 1. *For each non-stubborn node $i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$, there exists a directed path from node i to at least one stubborn node.*

This assumption imposes the mild restriction that every non-stubborn individual in the network can reach at least one stubborn agent through (possibly) a multi-hop path.

2.2. Effects of Stubborn Agents: From Consensus to Disagreement

The voter model without the stubborn agents was introduced independently in Clifford and Sudbury (1973), Holley and Liggett (1975). The convergence behavior of agent states generated with this model was studied extensively in the literature. In particular, it was shown in Aldous and Fill (1994) that agent states reach a consensus over any connected graph \mathcal{G} , *i.e.*,

$$\lim_{t \rightarrow \infty} \mathbb{P}(x_i(t) \neq x_j(t)) = 0 \quad \text{for all } i, j \in \mathcal{V}. \quad (2)$$

Liggett showed that this result can be extended to d -dimensional infinite lattices for $d = 1, 2$ (see Cox and Griffeath (1986), Holley and Liggett (1975), Liggett (2005)). For $d \geq 3$, he showed that $x(t)$ approaches a non-trivial equilibrium where the individual state values are neither independent nor fully correlated with each other.

In the voter model with stubborn agents, agent states do not necessarily reach a consensus since every agent is potentially (indirectly) connected to stubborn agents with distinct opinions which are never updated. Instead, we show in the following that the agent state vector converges in distribution, *i.e.*, there exists a random vector x^* such that

$$\lim_{t \rightarrow \infty} \mathbb{P}(x(t) \in \mathcal{A}) = \mathbb{P}(x^* \in \mathcal{A}), \quad \text{for all } \mathcal{A} \subset \mathbb{R}^N \text{ in the continuity set of } x^*.$$

In the rest of the paper, we will denote convergence in distribution as $x(t) \xrightarrow{D} x^*$. We refer to the probability distribution of the random vector x^* as the *stationary distribution* of agent state vector $x(t)$, and x_i^* as the *asymptotic opinion* of agent i .

THEOREM 1. *Let Assumption 1 hold. The agent state vector $x(t)$ has a unique stationary distribution.*

Our proof is based on the fact the stochastic process governing the evolution of $x(t)$ is a Markov process with a single recurrent class. We note that the unique stationary distribution of $x(t)$ is a function of the underlying graph \mathcal{G} and the stubborn sets $\mathcal{V}_0, \mathcal{V}_1$, and is independent of the initial opinions of the non-stubborn agents. The argument follows from the fact that the stationary distribution of $x(t)$ is unique, thus it is independent of the initial conditions. Hence, the initial opinions of the non-stubborn agents do not have any effect on the collective opinion of the society in the long run.

In the following, we will characterize the first two moments of the stationary distribution. However, we will first introduce the dual approach for the voter model with stubborn agents, which will play a key role in the subsequent analysis.

2.3. The Dual Approach for the Voter Model with Stubborn Agents

One of the key methods in the analysis of the voter model (without the stubborn agents) is the *dual approach*, where the agent state evolution is related to a dual process defined through *coalescing random walks* on the graph \mathcal{G} (see Diekmann et al. (1999), Holley and Liggett (1975), Liggett (2005) for more details). In the following, we first briefly describe the dual approach for the voter model and then show that a similar process can be defined even in the presence of stubborn agents.

2.3.1. The Dual Approach for the Original Voter Model The following discussion is based on material introduced in (Diekmann et al. 1999, Ch. 1.1). For a given agent $i \in \mathcal{V}$ and time instant $T \geq 0$, we define a *random walk* $\{Y_i^T(T-s) : s \in [0, T]\}$ on \mathcal{G} such that:

$$Y_i^T(T-s) = m,$$

if and only if there is a sequence of nodes $\{u_l\}_{l=0}^n$ with $u_0 = i, u_n = m$ and increasing sequence of times $\{r_l\}_{l=0}^{n+1}$ with $r_0 = 0, r_{n+1} = s$ and:

1. Agent $u_{l-1} \in \mathcal{V}$ has adopted the opinion of agent u_l at time $T - r_l$ for all $l \in \{1, \dots, n\}$.
2. Agent $u_{l-1} \in \mathcal{V}$ has not reassessed his opinion in $[T - r_{l-1}, T - r_l)$ for all $l \in \{1, \dots, n\}$.
3. Agent $u_n \in \mathcal{V}$ has not reassessed his opinion in $[T - r_n, T - r_{n+1}]$.

Using the definition, we can show that:

$$x_i(T) = x_{Y_i^T(T-s)}(s), \quad s \in [0, T]. \quad (3)$$

In other words, the opinion of agent i at time T is equal to the opinion of agent $Y_i^T(T-s)$ at time s . Therefore, the random walk $Y_i^T(T-s)$ tracks the source of the opinion of agent i at time T .

Figure 1(a) illustrates a sample update path for the voter model. In this example, we focus on a snapshot of the voter model process $t \in [0, T]$ on a line network with 4 nodes. The x-axis denotes different individuals in the network, and the y-axis denotes the continuous time interval. Whenever an agent adopts the opinion of one of his neighbors, it will be denoted by an arrow from that agent to his neighbor. For instance, agent 4 adopts the opinion of agent 3 at time instant t_1 , thus there exists an arrow directed from agent 4 to agent 3 at t_1 . If we focus on the process $Y_i^T(T-s)$ for $i = 1$, we observe that:

$$Y_1^T(T-s) = \begin{cases} 1 & \text{for } s \in [T, t_4), \\ 2 & \text{for } s \in [t_4, t_3), \\ 3 & \text{for } s \in [t_3, t_2), \\ 4 & \text{for } s \in [t_2, t_1), \\ 3 & \text{for } s \in [t_1, 0]. \end{cases}$$

Agent $i = 1$ does not reevaluate his opinion on the time interval $(t_4, T]$ according to Figure 1(a). This implies that $Y_1^T(T-s)$ is equal to 1 on $s \in [T, t_4)$. However, agent 1 adopts the opinion of

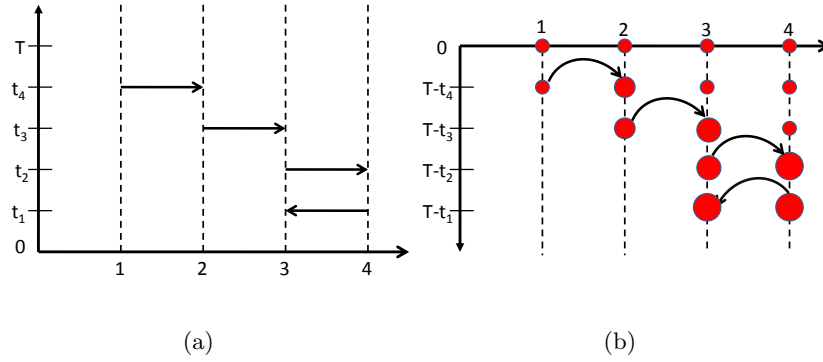


Figure 1 Sample paths for the voter model and the corresponding dual process.

agent 2 at time t_4 , therefore $Y_1^T(T-s)$ jumps to 2 at $s = t_4$, stays there until agent 2 adopts the opinion of agent 3 at time t_3 . Similarly, one can show that $Y_1^T(T) = 3$, thus $x_1(T) = x_3(0)$ by Eq. (3). In other words, the opinion of agent i at time instant T is equal to the opinion of agent 3 at time 0.

We note that the randomness in $Y_i^T(T-s)$ is due to the fact that agents adopt the opinions of randomly chosen neighbors at random times. Since each individual is realized according to a rate 1 Poisson process in the voter model, the random walk $Y_i^T(T-s)$ is also a rate 1 random walk on \mathcal{G} .

Moreover, if we want to track the origins of opinions of multiple agents, *i.e.*, at $\mathcal{B} \subset \mathcal{V}$, then we can define a joint process

$$Y_{\mathcal{B}}^T(T-s) = \{Y_i^T(T-s) | i \in \mathcal{B}\}, \quad (4)$$

such that:

- Two random walks $Y_i^T(T-s), Y_j^T(T-s)$ for any $i, j \in \mathcal{B}$ move independently until they hit.
- If two random walks meet at $r \in [0, T]$, then they will collide and move together, *i.e.*, $Y_i^T(T-s) = Y_j^T(T-s), s \in [0, r]$.

The crucial point is that if two or more of these processes reside at the same location at the same time, they will collide and move together, since the origin of the opinion at that node and time will be the same for both processes. Therefore, the joint process $Y_{\mathcal{B}}^T(T-s)$ defines *coalescing random walks* on graph \mathcal{G} , which will be referred as the *dual process* for the voter model.

The dual process for the sample path in Figure 1(a) is given in Figure 1(b). We note that at $s = t_1$ there exists a single walk on the network, *i.e.*, all 4 random walks have already coalesced into a single one and the joint walk resides at node 3. Therefore, agent states reach a consensus before time T in our example, and the consensus value is given by the initial opinion of agent 3.

2.3.2. The Dual Approach for the Voter Model with Stubborn Agents For the voter model with stubborn agents, we consider a similar dual process using coalescing random walks defined through Eq. (4). Note that, in the presence of stubborn agents, the stubborn nodes will be absorbing states for the processes $Y_i^T(T - s)$. In other words, if a sample path of the process hits one of the stubborn nodes $j \in \{\mathcal{V}_0 \cup \mathcal{V}_1\}$, it will be absorbed by that particular node, *i.e.*,

$$Y_i^T(T - s^*) = j, j \in \{\mathcal{V}_0 \cup \mathcal{V}_1\} \Rightarrow Y_i^T(T - s) = j, s \in [0, s^*].$$

The above argument follows from the fact that stubborn nodes do not adopt the opinions of their neighbors. Therefore, the dual process for the voter model with stubborn agents is the coalescing random walk with absorbing states $\{\mathcal{V}_0 \cup \mathcal{V}_1\}$.

Finally, we note that, for a given $T \geq 0$, $Y_i^T(T - s)$ is defined for $s \in [0, T]$. We define a generalized process $Y_i(T - s)$, which is defined for all $s \geq 0$, by requiring that $Y_i(T - s)$ and $Y_i^T(T - s)$ have the same distribution for any $T \geq 0$ on the interval $s \in [0, T]$. Similarly, one can define a generalized coalescing random walks $Y_{\mathcal{B}}(T - s)$ for any given subset $\mathcal{B} \subset \mathcal{V}$.

3. The Average Opinion of the Society

In many applications, such as political elections, we are interested in understanding the collective opinion of the society. Therefore, in the rest of the paper, we study the *average opinion* of the society, *i.e.*,

$$\bar{x}(t) = \frac{1}{N} \sum_{i \in \mathcal{V}} x_i(t). \quad (5)$$

The following is an immediate corollary of Theorem 1.

COROLLARY 1. *The average opinion of the society converges in distribution, i.e., there exists a random variable $\bar{x}^* \in \mathbb{R}$ such that:*

$$\bar{x}(t) \xrightarrow{D} \bar{x}^*,$$

In the rest of the paper, we refer to \bar{x}^* as the *asymptotic average opinion*. In the following, we will characterize the first two moments of the stationary distribution of the average opinion of the society. In particular, we will focus on the quantities $\mathbb{E}\{\bar{x}^*\}$ and $\sigma(\bar{x}^*) \triangleq \mathbb{E}\{(\bar{x}^*)^2\} - (\mathbb{E}\{\bar{x}^*\})^2$.

3.1. Expected Value of the Average Opinion

We first focus on the first moment of the stationary distribution of the average opinion, i.e., $\mathbb{E}\{\bar{x}^*\}$.

For a given graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and stubborn sets $\mathcal{V}_0, \mathcal{V}_1$, we define a random walk $Z(t), t \geq 0$, in terms of its transition rates as:

$$i \rightarrow j = \begin{cases} \frac{1}{|\mathcal{N}_i|} & \text{if } i \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, j \in \mathcal{N}_i, \\ -1 & \text{if } i \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, i = j, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

In other words, $Z(t)$ corresponds to a continuous time random walk on \mathcal{G} , where transitions at each node is uniform within its neighborhood and the sets \mathcal{V}_0 and \mathcal{V}_1 form two distinct absorbing classes. For a given $i \in \mathcal{V}$, we define p_i as the probability that the random walk $Z(t)$ initiated at node i is absorbed by the set \mathcal{V}_1 , i.e.,

$$p_i = \lim_{t \rightarrow \infty} \mathbb{P}(Z(t) \in \mathcal{V}_1 | Z(0) = i).$$

The following proposition characterizes the expected value of the asymptotic opinion of agent i .

PROPOSITION 1. *The expected value of the asymptotic opinion of agent $i \in \mathcal{V}$ is equal to the probability that the random walk $Z(t)$ initiated at node i is absorbed by the set \mathcal{V}_1 , i.e.:*

$$\mathbb{E}\{x_i^*\} = p_i.$$

Another way to interpret this result is the following: We focus on a scenario where stubborn nodes are *mavens* on a given subject (Gladwell (2002)). For instance, in the case of political elections with

two opponents, mavens are the individuals who are confident about their decisions (*left, right*). In the case of new product diffusion, mavens may have purchased the product already and assessed the true quality of the product (*high, low*). An individual i , who is not a maven, is seeking reliable information on the subject. He chooses one of his neighbors randomly and asks his opinion on the subject. If the selected neighbor is a maven, the neighbor replies back with his opinion, and individual i adopts it. Otherwise, the neighbor passes the query to one of his randomly selected neighbors. The message passing process continues until the message hits a maven, and individual i adopts that particular maven's opinion. In this particular scenario, the probability that individual i adopts maven j 's opinion is the frequency of messages originated from node i hitting j before hitting any other maven. Similarly, if there are two classes of mavens \mathcal{V}_0 and \mathcal{V}_1 , the probability that individual i adopts \mathcal{V}_1 's opinion is the frequency of messages originated from node i hitting \mathcal{V}_1 before hitting \mathcal{V}_0 , which, in the limit, is equal to p_i . Therefore, for the purposes of analyzing expected opinion in the voter model, one may interpret non-stubborn agents as individuals looking for information held by mavens (stubborn nodes) via naive random search.

We will refer to the expected value of the asymptotic opinion of an agent as the *bias of the agent*, since the quantity is an indicator of how biased that particular individual is towards the opinion of the stubborn set \mathcal{V}_1 . In particular, the higher the expected value is, the higher the bias is and the more favorable the agent is towards opinion 1. Similarly, we will refer to the expected value of the asymptotic average opinion as the *bias of the society*.

An immediate corollary of Proposition 1 is the following.

COROLLARY 2. *The expected value of the asymptotic average opinion in Eq. (5) is equal to:*

$$\mathbb{E}\{\bar{x}^*\} = \frac{1}{N} \sum_{i \in \mathcal{V}} p_i. \quad (7)$$

The above corollary characterizes the bias of the society in terms of the absorption probabilities of the random walk in Eq. (6). We emphasize that these probabilities are functions of the graph structure \mathcal{G} as well as the locations of the stubborn sets \mathcal{V}_0 and \mathcal{V}_1 .

3.2. Variance of the Average Opinion

In the previous subsection, we have characterized the bias of the society which measures how favorable the society is towards opinion 1. This measure provides us information about the long run average behavior of the society. However, since it is in expectation, it does not capture the actual behavior of the society at a particular time t . We characterize the fluctuations around the expectation using the variance of the average opinion.

We first focus on the limit of the variance of the average opinion:

$$\begin{aligned} \lim_{t \rightarrow \infty} \sigma(\bar{x}(t)) &= \lim_{t \rightarrow \infty} \mathbb{E} \left\{ (\bar{x}(t) - \mathbb{E} \{\bar{x}(t)\})^2 \right\} \\ &= \frac{1}{N^2} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} \lim_{t \rightarrow \infty} (\mathbb{E} \{x_i(t)x_j(t)\} - \mathbb{E} \{x_i(t)\} \mathbb{E} \{x_j(t)\}), \end{aligned} \quad (8)$$

where $\sigma(\cdot)$ denotes the variance of its argument. Since we have already characterized $\lim_{t \rightarrow \infty} \mathbb{E} \{x_i(t)\}$ terms as the bias of individual i in Proposition 1, we will now focus on the term $\lim_{t \rightarrow \infty} \mathbb{E} \{x_i(t)x_j(t)\}$. In particular, we will show that $\lim_{t \rightarrow \infty} \mathbb{E} \{x_i(t)x_j(t)\}$ exists, for all $i, j \in \mathcal{V}$ and provide an explicit characterization for its value. We then use this characterization to provide bounds on the variance as a function of the graph properties.

Since $x_i(t)$ is a binary random variable, for all $i \in \mathcal{V}$ and all $t \geq 0$, we have:

$$\lim_{t \rightarrow \infty} \mathbb{E} \{x_i(t)x_j(t)\} = \lim_{t \rightarrow \infty} \mathbb{P}(x_i(t) = x_j(t) = 1). \quad (9)$$

Therefore, this particular term is equal to the probability that the opinions of agent i and agent j are both equal to one in the limit. Due to potential correlation between the values of agent i and j , one cannot estimate $\lim_{t \rightarrow \infty} \mathbb{P}(x_i(t) = x_j(t) = 1)$ in terms of individual expectations. In the following, we will completely characterize these terms by using the dual approach that we have introduced in Section 2.3.2.

As we have discussed in Section 2.3, for a given agent pair i, j , by choosing $s = 0$ in Eq. (4), we have:

$$(x_i(t), x_j(t)) = x_{Y_{\{i,j\}}(t)}(0), \quad t \geq 0,$$

where $Y_{\{i,j\}}(t)$ is the coalescing random walks initiated at nodes i and j , respectively. We note that individual processes $Y_i(t)$ and $Y_j(t)$ are both random walks on \mathcal{G} with transition rates given in Eq. (6). Since \mathcal{G} is a finite size graph, the connectivity of \mathcal{G} is such that there exists a path from each non-stubborn node to at least one stubborn node, and there exist two nonempty absorbing classes \mathcal{V}_0 and \mathcal{V}_1 , then $\lim_{t \rightarrow \infty} Y_{\{i,j\}}(t) \in \{\mathcal{V}_0 \cup \mathcal{V}_1\} \times \{\mathcal{V}_0 \cup \mathcal{V}_1\}$ with probability 1 (Seneta 1981, Ch. 4.2)). By noting that $x_j(0) = 0$ if $j \in \mathcal{V}_0$, and $x_j(0) = 1$ if $j \in \mathcal{V}_1$,

$$\lim_{t \rightarrow \infty} \mathbb{P}(x_i(t) = x_j(t) = 1) = \lim_{t \rightarrow \infty} \mathbb{P}(Y_{\{i,j\}}(t) \in \{\mathcal{V}_1 \times \mathcal{V}_1\}). \quad (10)$$

In other words, the probability that the opinions of agents i and j are both equal to one is equal to the probability that the coalescing random walks initiated at node i and node j are both absorbed by the stubborn set \mathcal{V}_1 .

To calculate this probability, we construct a product graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$, where $\mathcal{V}' = \mathcal{V} \times \mathcal{V}$. We denote each element of \mathcal{V}' as (i, j) where i and j correspond to node indices in the original graph \mathcal{G} . We note that the cardinality of the set \mathcal{V}' is N^2 . Moreover, there exists an edge between two nodes (i, j) and (m, n) , if one of the following holds:

- $i = m$, and there exists an edge between nodes j and n in the original graph \mathcal{G} , *i.e.*, $(j, n) \in \mathcal{E}$.
- $j = n$, and there exists an edge between nodes i and m in the original graph \mathcal{G} , *i.e.*, $(i, m) \in \mathcal{E}$.
- $i = j$, and $m = n$, and there exists an edge between nodes i and m in the original graph \mathcal{G} , *i.e.*, $(i, m) \in \mathcal{E}$.

On this particular graph \mathcal{G}' , we define a continuous time random walk $Z'(t), t \geq 0$ with transition rates:

$$(i, j) \rightarrow (m, n) = \begin{cases} \frac{1}{|\mathcal{N}_i|} & \text{if } i \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, i \neq j, n = j, m \in \mathcal{N}_i, \\ \frac{1}{|\mathcal{N}_j|} & \text{if } j \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, i \neq j, m = i, n \in \mathcal{N}_j, \\ \frac{1}{|\mathcal{N}_i|} & \text{if } i \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, m = n, i = j, m \in \mathcal{N}_i, \\ -2 & \text{if } i = m, j = n, i \neq j, i, j \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, \\ -1 & \text{if } i = m = j = n, i \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, \\ -1 & \text{if } i = m, j = n, i \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, j \in \{\mathcal{V}_0 \cup \mathcal{V}_1\} \text{ or } j \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, i \in \{\mathcal{V}_0 \cup \mathcal{V}_1\}, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

The random walk in Eq. (11) is equivalent to the dual process $Y_{\{i,j\}}(t)$. To see this, we observe that the transition rates of the process $Z'(t)$ are such that two particles jumps independently until they coalesce and move together once they collide, *i.e.*, $m = n$, and sets $\mathcal{V}_0, \mathcal{V}_1$ form absorbing sets

By combining our discussion above and Eq. (9), we conclude the following:

LEMMA 1. *For any $i, j \in \mathcal{G}$, the limiting covariance between the opinions of these agents is given by*

$$\begin{aligned} \lim_{t \rightarrow \infty} (\mathbb{E} \{x_i(t)x_j(t)\} - \mathbb{E} \{x_i(t)\} \mathbb{E} \{x_j(t)\}) &= \lim_{t \rightarrow \infty} \mathbb{P}(x_i(t) = x_j(t) = 1) - p_i p_j \\ &= \sum_{m, n \in \mathcal{V}_1} \lim_{t \rightarrow \infty} \mathbb{P}(Z'(t) = (m, n) | Z'(0) = (i, j)) - p_i p_j, \end{aligned}$$

where $Z'(t), t \geq 0$ is the random walk on \mathcal{G}' whose transition rates are given in Eq. (11).

This lemma characterizes the individual terms in Eq. (8), thus provides a characterization of the limit of the variance. The next proposition introduces the variance of the asymptotic average opinion, *i.e.*, $\sigma(\bar{x}^*)$.

PROPOSITION 2. *The variance of the asymptotic average opinion is equal to:*

$$\sigma(\bar{x}^*) = \frac{1}{N^2} \sum_{i, j \in \mathcal{V}} \left(\sum_{m, n \in \mathcal{V}_1} \lim_{t \rightarrow \infty} \mathbb{P}(Z'(t) = (m, n) | Z'(0) = (i, j)) - p_i p_j \right).$$

While the above result provides an explicit characterization of the variance, it is in terms of the absorption probabilities of $Z'(t)$ on \mathcal{G}' , and does not provide intuition on the effect of the network structure \mathcal{G} on the variance. As a first step in understanding this dependence, we will first show that the variance can be written as a system of harmonic equations on the graph \mathcal{G} :

PROPOSITION 3. *The variance of the asymptotic average opinion is equal to:*

$$\sigma(\bar{x}^*) = \frac{1}{N^2} \sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} (p_l - p_l^2) \sum_{i, j \in \mathcal{V}} \mathbb{P}(I_{ij}^l), \quad (12)$$

where, for each $l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$, $\{I_{ij}^l\}_{i, j \in \mathcal{V}}$ is the event that the random walk $Z'(t)$ initialized at (i, j) hits (l, l) before hitting any other node of the form (l', l') or being absorbed by the absorbing set, and it is the solution of the following system of harmonic equations:

$$\mathbb{P}(I_{ij}^l) = \begin{cases} 0 & \text{if } i = j \neq l, \text{ or } i \in \{\mathcal{V}_0 \cup \mathcal{V}_1\}, \text{ or } j \in \{\mathcal{V}_0 \cup \mathcal{V}_1\}, \\ 1 & \text{if } i = j = l, l \notin \{\mathcal{V}_0 \cup \mathcal{V}_1\}, \\ \frac{1}{2|\mathcal{N}_i|} \sum_{i' \in \mathcal{N}_i} \mathbb{P}(I_{i'j}^l) + \frac{1}{2|\mathcal{N}_j|} \sum_{j' \in \mathcal{N}_j} \mathbb{P}(I_{ij'}^l) & \text{otherwise.} \end{cases}$$

While Proposition 3 provides an analytical way to calculate the exact variance, it still falls short of providing intuition in terms of the graph characteristics and the location of the stubborn agent sets. In the following, we will introduce an upper bound for the variance in terms of the properties of the original graph \mathcal{G} as well as locations of the stubborn agents. Before stating our result, we first define a new variable r_i as follows: Define a random walk on the graph \mathcal{G} denoted by $Z_i(t)$ where node $i \in \mathcal{V} \setminus (\mathcal{V}_0 \cup \mathcal{V}_1)$ forms a new stubborn class \mathcal{V}_2 . For a given node $j \in \mathcal{V}$, define \tilde{p}_j^i as the probability that the random walk $Z_i(t)$ initialized at node j is absorbed by the class \mathcal{V}_2 , *i.e.*, node i , before hitting classes $\mathcal{V}_0, \mathcal{V}_1$. Define

$$r^i \triangleq \sum_{j \in \mathcal{V}} \tilde{p}_j^i. \quad (13)$$

Hence, r^i provides a relative measure of centrality for node i compared to the rest of the stubborn agents. Note that $r^i \geq 1$ for all i since $\tilde{p}_i^i = 1$ for all i .

Our main result provides a bound on the variance of the asymptotic average opinion in terms of this centrality measure.

THEOREM 2. *The variance of the asymptotic average opinion is bounded by:*

$$\sigma(\bar{x}^*) \leq \frac{1}{N^2} \sum_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} (p_i - p_i^2)(r^i)^2. \quad (14)$$

The lemma provides a bound on the variance in terms of two quantities: The first is the term $p_i - p_i^2$ for a given agent i . Since p_i is the bias of agent i and agent values are binary random variables, $p_i - p_i^2$ is the variance of the asymptotic opinion of node i , *i.e.*, $\sigma(x_i^*)$. This particular term is maximized when $p_i = 0.5$, *i.e.*, when individual i is equally biased towards 0 and 1. In view of the correlation among the agent states, the variance in the opinion of agent i has a broader effect on the network, which is reflected in the scaling by $(r^i)^2$. We note that r^i is a centrality measure for agent i given by the (sum) probability that random walks initiated at nodes in the network are absorbed by node i before hitting the stubborn sets $\mathcal{V}_0, \mathcal{V}_1$.

We can motivate the relevance of the measure r^i using the intuition of random search for information held by stubborn agents, who act as mavens (see the discussion following Proposition 1).

Agent i in addition to sets \mathcal{V}_0 and \mathcal{V}_1 acts as a maven, and r^i is the (sum) frequency of queries that hit maven i before hitting mavens $\mathcal{V}_0, \mathcal{V}_1$. We will refer to r^i as the *relative influence* of node i in the network (with respect stubborn sets $\mathcal{V}_0, \mathcal{V}_1$), since it is a measure of how many queries it can attract in the presence of stubborn sets $\mathcal{V}_0, \mathcal{V}_1$.

The upper bound in Theorem 2 suggests that the variance of the average opinion of the society may be higher when individuals with high relative influence are equally biased towards 0 and 1, *i.e.*, they have high variability in their opinions. This is intuitive since the variance in the opinions of such agents will spread to a more significant portion of the network. Moreover, the locations of the agents with high relative influence as well as the uncertainty in their decisions depend on both the graph \mathcal{G} and the stubborn sets $\mathcal{V}_0, \mathcal{V}_1$. Therefore, the graph \mathcal{G} can not, by itself, determine the variability in society's bias; it is the combination of the graph \mathcal{G} and the locations of the stubborn agents $\mathcal{V}_0, \mathcal{V}_1$ that determine the variance. We refer to societies with low (high) variances as *stable* (*unstable*).

Figure 2 provides a stark example that illustrates the effect of the location of stubborn agents on the asymptotic variance of the average opinion. In particular, we consider two networks with different stubborn set configurations. In both networks there are N nodes, two on the sides and $N - 2$ in the middle. Stubborn nodes are indicated by 0 and 1, respectively. The rest of the nodes in the network are assumed to be non-stubborn. We note that both networks have the same underlying graph \mathcal{G} while the locations of the stubborn sets are different.

In the first configuration, in view of the symmetry of the graph with respect to \mathcal{V}_0 and \mathcal{V}_1 , each non-stubborn node has bias equal to 0.5. For a given non-stubborn node i , its relative influence with respect to the stubborn sets is small. In particular, $r^i = 1$ for all $i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$. It follows from the fact that for a given non-stubborn agent i , $\tilde{p}_i^i = 1$, and $\tilde{p}_j^i = 0$, for $j \neq i$.

In the second configuration, similar to the previous case, all non-stubborn agents have biases equal to 0.5 due to symmetry. However, one can show that the relative influence of a given non-stubborn node i is $\Theta(N)$. This follows from the fact that, when it acts as a maven, each non-stubborn node becomes as attractive as the stubborn agents.

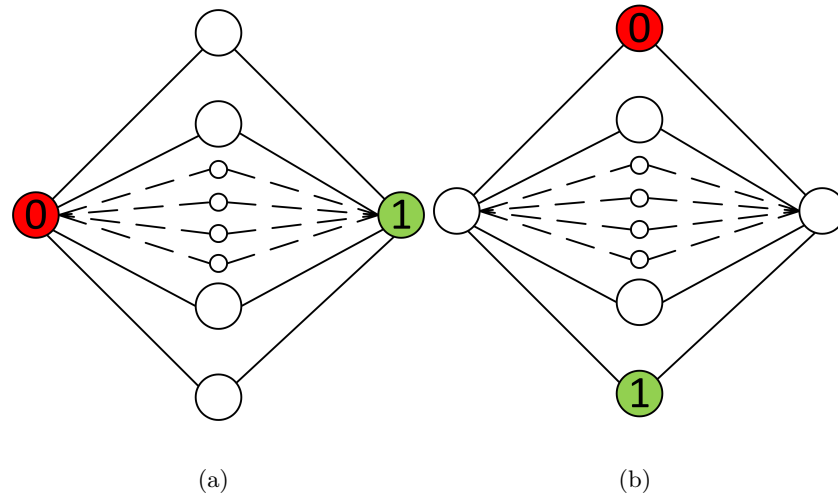


Figure 2 Two different configurations of stubborn agents on the same graph \mathcal{G} .

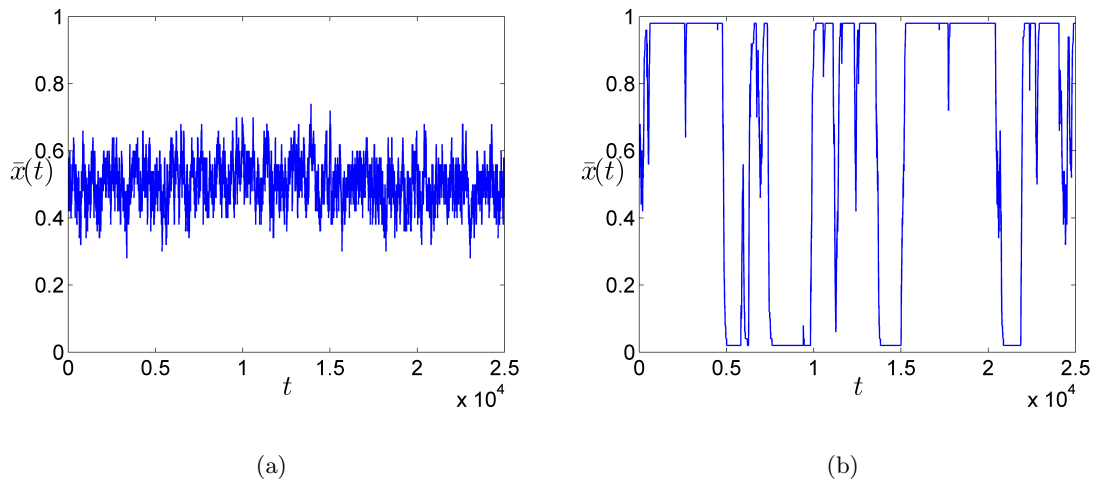


Figure 3 Sample paths of the average opinions for configurations in Figures 2(a) and 2(b).

Figure 3 show the impact of a local modification in the location of stubborn agents on the variance of the average opinion. Figures 3(a) and 3(b) show sample paths of the average opinion for the configurations introduced in Figures 2(a) and 2(b), respectively. In both configurations, $N = 100$ and the initial opinions of non-stubborn agents are drawn uniformly randomly. Parallel to our discussions above, while the mean value of the average opinion is 0.5 for both configurations, the fluctuations in the latter configuration is significantly larger than the former one.

4. Examples

In this section, we will focus on two special graphs, and analyze the behavior of our model on these networks in detail.

4.1. Complete Graph

This type of graph corresponds to a society where every single individual knows every other individual, *i.e.*, $\mathcal{N}_i = \mathcal{V}$ for all $i \in \mathcal{V}$. Such connectivity has been argued to represent village-like communities (Easley and Kleinberg (2010)). For given stubborn sets $\mathcal{V}_0, \mathcal{V}_1$ and network size N , the following holds:

$$\mathbb{E}\{\bar{x}^*\} = \frac{|\mathcal{V}_1|}{|\mathcal{V}_1| + |\mathcal{V}_0|}, \quad (15)$$

$$\sigma(\bar{x}^*) = \left(1 - \frac{|\mathcal{V}_1| + |\mathcal{V}_0|}{N}\right) \frac{|\mathcal{V}_1||\mathcal{V}_0|}{(|\mathcal{V}_1| + |\mathcal{V}_0|)^2} \frac{1}{1 + |\mathcal{V}_1| + |\mathcal{V}_0|}. \quad (16)$$

These results follow from basic Markov Chain analysis and proofs are omitted due to lack of space. We first note that identities given above do not depend on the location of the stubborn sets since the underlying connectivity is homogenous. The average bias of the society is simply the percentage of the stubborn agents of type 1 Eq. (15). Since the underlying network is a complete graph, absorption probabilities increase linearly with the size of the stubborn set \mathcal{V}_1 .

Assuming $|\mathcal{V}_1| + |\mathcal{V}_2| = O(1)$, the variance of the society in Eq. (16) converges a constant as N gets large, *i.e.*

$$\sigma(\bar{x}^*) \approx \frac{|\mathcal{V}_1||\mathcal{V}_0|}{(|\mathcal{V}_1| + |\mathcal{V}_0|)^2} \frac{1}{1 + |\mathcal{V}_1| + |\mathcal{V}_0|}$$

We note that unless \mathcal{V}_0 or \mathcal{V}_1 is an empty set, the constant will be non-zero. Therefore, the variance of the society can not go to zero (as size of the graph increases) if the total number of stubborn agents is fixed. However, as long as $|\mathcal{V}_0| + |\mathcal{V}_1| = \omega(1)$, then the variance approaches 0 as N gets large. Therefore, regardless of how slow it is, an increase in the number of stubborn people, results in highly stable societies in the case of fully connected networks. We note that our analysis for the complete graphs coincides with Mobilia *et al.*'s results which have been derived using mean field

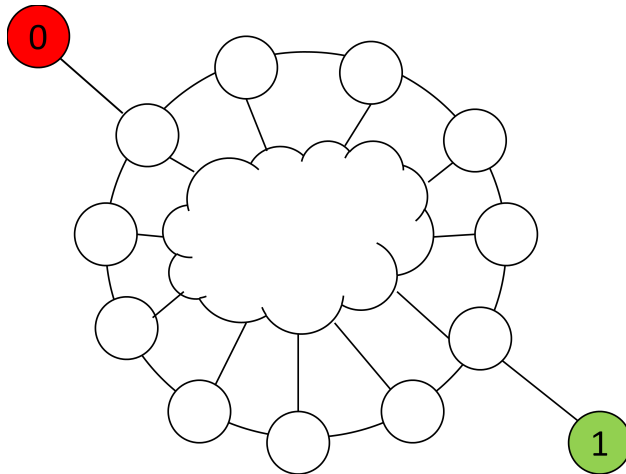


Figure 4 A graph where non-stubborn agents form a complete graph and each stubborn is connected to a single non-stubborn agent.

approximation (Mobilia et al. (2007)). Such a result is not surprising since mean field analysis is tight on homogenous graphs. A slight modification of the complete graph is given in Figure 4. In this setting, non-stubborn nodes form a complete graph while each stubborn node is connected to a single non-stubborn node. Such connectivity might represent a scenario where stubborn agents can not directly communicate with the rest of the society, but through their *middle men* and the rest of the network is highly connected. One can show with some algebra that:

$$\mathbb{E}\{\bar{x}^*\} = 0.5,$$

$$\sigma(\bar{x}^*) = \frac{(N+2)^5 + 2(N+2)^4 - 6(N+2)^2 - 5(N+2) + 2}{8(N+2)^2(N+3)^3}.$$

We can compare these two quantities with their counterparts in Eqs. (15) and (16) by choosing $|\mathcal{V}_0| = |\mathcal{V}_1| = 1$. We observe that while the bias has not changed, the variance in the latter setting is larger than in the former one. In fact, for large enough N , the ratio of two variances approaches $3/2$. Such an observation is intuitive since, in the latter scenario, the relative influences of the non-stubborn nodes increases (stubborn nodes are outside of the fully connected part of the society) although individual variances stay constant.

4.2. Regular Graphs

In this section, we focus on d -regular graphs where each agent in the network has exactly d neighbors. Moreover, we assume that $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$, *i.e.*, the edges are symmetric. We note that this particular assumption implies reversibility of the dual random walk. We will use the analogy of a graph \mathcal{G} to an electrical network and provide a bound using the effective resistance measure in electrical networks (see (Levin et al. 2006, Ch. 2) for a detailed discussion of electrical network representation of graphs and their relation to reversible Markov chains). In particular, for a given graph \mathcal{G} , we define an electrical network with node set and edge set coinciding with that of \mathcal{G} . We assume that each edge $(i, j) \in \mathcal{E}$ has a unit resistance. We assume that voltages $W(i)$ and $W(j)$ are applied the nodes $i, j \in \mathcal{V}$, which induce a current flow C in the network. The effective resistance between nodes i and j is defined as

$$R(i \leftrightarrow j) = \frac{W(i) - W(j)}{|C|}.$$

We next state the main result of the section.

PROPOSITION 4. *For a given d regular network $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with stubborn sets $\mathcal{V}_0, \mathcal{V}_1$, the variance of the asymptotic average opinion is bounded by,*

$$\sigma(\bar{x}^*) \leq \frac{d}{4N^2} \sum_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} R(i \leftrightarrow (\mathcal{V}_0 \cup \mathcal{V}_1)) r^i. \quad (17)$$

While the upper bound in Eq. (14) holds for all topologies and provides significant intuition, Eq. (17) will be tighter for certain configurations. For instance, for a complete graph (which is an N -regular graph), we can use the relation that (Levin et al. (2006)):

$$dR(i \leftrightarrow (\mathcal{V}_0 \cup \mathcal{V}_1)) = \frac{1}{\mathbb{P}_j(T_{\mathcal{V}_0 \cup \mathcal{V}_1} < T_j^{1+})},$$

(where $\mathbb{P}_j(T_{\mathcal{V}_0 \cup \mathcal{V}_1} < T_j^{1+})$ is the probability that random walk initiated at node j hits the set $\mathcal{V}_0 \cup \mathcal{V}_1$ before returning itself) to show that

$$dR(i \leftrightarrow (\mathcal{V}_0 \cup \mathcal{V}_1)) = \frac{N(|\mathcal{V}_0| + |\mathcal{V}_1| + 1)}{(N + 1)(|\mathcal{V}_0| + |\mathcal{V}_1|)}.$$

Therefore the ratio of the upper bound in Proposition 4 to the exact variance given in Eq. (16) is equal to:

$$\frac{N}{2(N+1)} \left(\frac{|\mathcal{V}_1|}{|\mathcal{V}_0|} + \frac{|\mathcal{V}_0|}{|\mathcal{V}_1|} + 1 \right).$$

We note that the upper bound is tight when the cardinality of the opposing stubborn sets are in the same order.

5. A Measure for Convergence time

In the previous sections, we have provided bounds on the moments of the stationary distribution of the average opinion of the society. In this section, we present a measure of convergence time for agent opinions and provide bounds on this measure.

In Theorem 1 of Section 2, we have shown that the stationary distribution of $x(t)$ is only a function of \mathcal{G} and $\mathcal{V}_0, \mathcal{V}_1$, thus is independent of the initial opinion distribution of the non-stubborn agents. Using this fact, we propose a convergence time measure $\mathbb{E}\{T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}$, where $T_{\mathcal{V}_0 \cup \mathcal{V}_1}$ is the first time when the opinions of the stubborn agents diffuse to the whole graph \mathcal{G} and the opinion state vector $x(T_{\mathcal{V}_0 \cup \mathcal{V}_1})$ becomes independent of the initial opinions of the non-stubborn individuals.

We can define $T_{\mathcal{V}_0 \cup \mathcal{V}_1}$ as:

$$T_{\mathcal{V}_0 \cup \mathcal{V}_1} = \arg \min_{t \geq 0} \{Y_{\mathcal{V}}(t) \in \{\mathcal{V}_0 \cup \mathcal{V}_1\}^N\},$$

where $Y_{\mathcal{V}}(t)$ is the dual process defined in Eq. (4) with $\mathcal{B} = \mathcal{V}$. In other words, $T_{\mathcal{V}_0 \cup \mathcal{V}_1}$ is the first time that the source of all the agent opinions in the network is the stubborn set, *i.e.*, the coalescing random walks initiated at all nodes in \mathcal{G} are all absorbed by the set $\{\mathcal{V}_0 \cup \mathcal{V}_1\}$.

At this point, we are ready to introduce the main result of the section:

PROPOSITION 5. *The convergence time measure $\mathbb{E}\{T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}$ is upper bounded by:*

$$\mathbb{E}\{T_{\mathcal{V}_0 \cup \mathcal{V}_1}\} \leq e(2 + (N - |\mathcal{V}_0 \cup \mathcal{V}_1|)) \max_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{E}_i T_{\mathcal{V}_0 \cup \mathcal{V}_1} \quad (18)$$

where $\mathbb{E}_i T_{\mathcal{V}_0 \cup \mathcal{V}_1}$ is the expected first hitting time on the set $\{\mathcal{V}_0 \cup \mathcal{V}_1\}$, when the random walk $Z(t)$ is initialized at node i .

Assuming the edges on \mathcal{G} are symmetric, and by using bounds on the expected hitting times on random walks on general graphs (see for example Table 2 of (Aldous and Fill 1994, Ch. 5.2)), Proposition 5 yields the following estimates on the convergence time measure $\mathbb{E}\{T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}$ for different network topologies: $\mathbb{E}\{T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}$ is $O(N \log(N - |\mathcal{V}_0 \cup \mathcal{V}_1|))$ for complete and dense regular graphs, $O(N^3 \log(N - |\mathcal{V}_0 \cup \mathcal{V}_1|))$ barbell topologies, and $O(N^2 \log(N - |\mathcal{V}_0 \cup \mathcal{V}_1|))$ for cycles. We note that these bounds are the worst case bounds in terms of the locations of the stubborn sets. Moreover, $O(N^3 \log(N - |\mathcal{V}_0 \cup \mathcal{V}_1|))$ is the bound for the worst case scenario, *i.e.*, is satisfied by any graph \mathcal{G} and stubborn sets $\mathcal{V}_0, \mathcal{V}_1$.

6. Optimal Stubborn Agent Placement

In Sections 3 and 4, we have characterized the long term behavior of the society in terms of the underlying connectivity and the locations of the stubborn sets. In this section, we will focus on a design problem which will be referred to as the *optimal stubborn agent placement* problem. In particular, we assume that we are given a network \mathcal{G} and a set of stubborn agents of type zero \mathcal{V}_0 with known locations. Given $k > 0$, our goal is to choose k nodes from the set $\mathcal{V} \setminus \{\mathcal{V}_0\}$ which will form the set \mathcal{V}_1 such that the bias of the society is maximized.

In the following, we will first focus on the case where $k = 1$, and then introduce the general formulation.

6.1. A special case: k=1

In this special setting, our goal is to pick a single node as a stubborn agent of type 1 given the sets $\mathcal{V}, \mathcal{V}_0$. We first recall that, by Proposition 1, the average opinion of a given node i is equal to the probability that the random walk $Z(t)$ is absorbed by \mathcal{V}_1 , which in this case is a single node l , given that the walk has been started at node i . Denoting this particular probability as $\tilde{p}_i^{\mathcal{V}_1}$, or equivalently \tilde{p}_i^l , our maximization problem can be written as:

$$\max_{l \in \mathcal{V} \setminus \mathcal{V}_0} \sum_{i \in \mathcal{V}} \tilde{p}_i^l. \quad (19)$$

We note that for a fixed l , the summation $\sum_{i \in \mathcal{V}} \tilde{p}_i^l$ is equal to r^l by Eq. (13).

Without loss of generality, we assume that $\{1, \dots, |\mathcal{V}_0|\}$ nodes are stubborn agents with type 0. Then, the transition probability matrix of the discrete time random walk on that particular graph (stubborn agents with type 0 being absorbing states) can be written as:

$$T = \begin{bmatrix} I_{\mathcal{V}_0} & 0 \\ C & D \end{bmatrix}. \quad (20)$$

where $I_{\mathcal{V}_0}$ is $|\mathcal{V}_0| \times |\mathcal{V}_0|$ identity matrix, $D \in \mathbb{R}^{(N-|\mathcal{V}_0|) \times (N-|\mathcal{V}_0|)}$. We note that C defines the transition probabilities from the non-stubborn nodes to the stubborn ones, and D determines the transition probabilities among non-stubborn nodes. The matrix $(I - D)^{-1}$ is the fundamental matrix of the chain and $[(I - D)^{-1}]_{ij}$ is equal to the expected number visits to node j before absorption, given that the walk has been initiated at node i ((Seneta 1981, Theorem 4.5)). Moreover, $(I - D)^{-1}$ can be determined completely for given \mathcal{V} and \mathcal{V}_0 , and is independent of the specific choice of \mathcal{V}_1 .

Without loss of generality, let's assume that we pick node $|\mathcal{V}_0| + 1$ as a candidate for our maximization problem in Eq. (19). Assigning the type 1 label to this particular node, the transition probability matrix of the random walk with an additional absorbing state $|\mathcal{V}_0| + 1$ becomes:

$$\tilde{T} = \begin{bmatrix} I_{\mathcal{V}_0} & 0 & 0 \\ 0 & 1 & 0 \\ \tilde{C} & c & \tilde{D} \end{bmatrix}, \quad (21)$$

where $\tilde{C} \in \mathbb{R}^{(N-|\mathcal{V}_0|-1) \times |\mathcal{V}_0|}$ defines the transition probabilities from non-stubborn nodes to the set \mathcal{V}_0 , $c \in \mathbb{R}^{(N-|\mathcal{V}_0|-1) \times 1}$ determines the transition probabilities from non-stubborn nodes to the candidate node, and \tilde{D} defines the transition probabilities among the non-stubborn nodes. By the theory of homogenous Markov Chains, $[\lim_{k \rightarrow \infty} (\tilde{T})^k]_{ij}$ corresponds to the probability that random walk initiated at node i is absorbed by node j ((Seneta 1981, Ch. 4.2)). With some algebra, one can show that:

$$\lim_{k \rightarrow \infty} (\tilde{T})^k = \lim_{k \rightarrow \infty} \begin{bmatrix} I_{\mathcal{V}_0} & 0 & 0 \\ 0 & 1 & 0 \\ \left(\sum_{l=1}^k \tilde{D}^l\right) \tilde{C} & \left(\sum_{l=1}^k \tilde{D}^l\right) c & \tilde{D}^k \end{bmatrix} = \begin{bmatrix} I_{\mathcal{V}_0} & 0 & 0 \\ 0 & 1 & 0 \\ (I - \tilde{D})^{-1} \tilde{C} & (I - \tilde{D})^{-1} c & 0 \end{bmatrix}. \quad (22)$$

It should be clear from Eq. (22) that the sum of the absorption probabilities by node $|\mathcal{V}_0| + 1$ is simply,

$$\sum_{i \in \mathcal{V}} p_i^{|\mathcal{V}_0|+1} = \mathbf{1}'(I - \tilde{D})^{-1}c, \quad (23)$$

where $\mathbf{1}$ is all ones vector with appropriate dimensions. In other words, if node $i = |\mathcal{V}_0| + 1$ is chosen as the stubborn agent with label one, then the value of the objective function in Eq. (19) is equal to Eq. (23).

We would like to emphasize the relationship between D in Eq. (20) and \tilde{D} in Eq. (21). Indeed,

$$D = \begin{bmatrix} 0 & v' \\ c & \tilde{D} \end{bmatrix}$$

where $c \in \mathbb{R}^{(N-|\mathcal{V}_0|-1) \times 1}$ governs the transition probabilities from non-stubborn nodes to the candidate node, and $v' \in \mathbb{R}^{1 \times (N-|\mathcal{V}_0|-1)}$ defines the transition probabilities from the candidate node to the non-stubborn agents. Thus, by using matrix inversion in block form, the fundamental matrix of T which is equal to $(I - D)^{-1}$ can be written as:

$$(I - D)^{-1} = \begin{bmatrix} 1 & -v' \\ -c & I - \tilde{D} \end{bmatrix}^{-1} = \begin{bmatrix} \gamma & \gamma v'(I - \tilde{D})^{-1} \\ \gamma(I - \tilde{D})^{-1}c & \gamma(I - \tilde{D})^{-1}c v'(I - \tilde{D})^{-1} + \gamma(I - \tilde{D})^{-1} \end{bmatrix},$$

where $\gamma = 1 - v'(I - D)^{-1}c$ is a scalar value. Note that the first column sum (the column corresponding to the specific choice of type 1 agent) of the fundamental matrix is simply $\gamma(1 + \mathbf{1}'(I - \tilde{D})^{-1}c)$. This is exactly γ times the value in Eq. (23), *i.e.*, γ times the value of the objective function of the optimization problem when node $i = |\mathcal{V}_0| + 1$ is chosen.

Repeating the argument above for $l \in \{|\mathcal{V}_0| + 1, \dots, \mathcal{V}\}$, we can state that, for $l \in \mathcal{V} \setminus \mathcal{V}_0$:

$$\sum_{i \in \mathcal{V}} \tilde{p}_i^l = \frac{\mathbf{1}'[(I - D)^{-1}]_{l-|\mathcal{V}_0|}}{[(I - D)^{-1}]_{l-|\mathcal{V}_0|, l-|\mathcal{V}_0|}},$$

where $[\cdot]_l$ denotes the l -th row of its argument. This particular relationship is interesting since the value of the objective function is dependent on the elements of the $(I - D)^{-1}$ matrix in Eq. (20) whose structure is independent of the particular choice l . The structure of the fundamental matrix is a function of the location of the stubborn agents with type 0 only. However, the dependence on the particular l is introduced through the selection of certain rows of the matrix. For algorithmic purposes, the fundamental matrix of the chain can be calculated once, and an exhaustive search over all the columns can be conducted for determining optimum l . The complexity of calculating the fundamental matrix is $O((N - |\mathcal{V}_0|)^3)$, since it requires the inversion of an $(N - |\mathcal{V}_0|) \times (N - |\mathcal{V}_0|)$

matrix. Once the fundamental matrix is computed, an exhaustive search over the columns can be conducted for determining the optimal l , bringing the total complexity to $O((N - |\mathcal{V}_0|)^5)$. In the following corollary, we summarize the main result of the subsection:

COROLLARY 3. *Given a network \mathcal{G} with the set of stubborn agents of type 0, \mathcal{V}_0 , for $k = 1$, the solution for the optimal agent placement problem is*

$$\arg \max_{l \in \mathcal{V} \setminus \mathcal{V}_0} \frac{\mathbf{1}'[(I - D)^{-1}]_{l-|\mathcal{V}_0|}}{[(I - D)^{-1}]_{l-|\mathcal{V}_0|, l-|\mathcal{V}_0|}},$$

where $(I - D)^{-1}$ is the fundamental matrix of the random walk on \mathcal{G} with absorbing set of states \mathcal{V}_0 .

The optimal stubborn agent placement problem for the voter model has been briefly discussed by Wu *et al.* in Wu and Huberman (2004). The authors concluded that nodes with higher degrees (number of neighbors) contribute more to the bias of the society regardless of the graph \mathcal{G} and the locations of the other stubborn nodes (Wu and Huberman (2004)). Therefore, the node with the highest degree is the solution for the optimal stubborn agent placement problem for $k = 1$. We would like to emphasize that their analysis is based on the mean field approximation technique as well as averaging over random graphs. On the other hand, our result in Corollary 3 is exact and shows that the optimal placement problem depends on the global properties of \mathcal{G} through the fundamental matrix D , thus the solution can not be determined only by the local neighborhood structures.

6.2. General case: $k > 1$

For any $k > 1$, our optimization problem becomes:

$$\max_{l_1, l_2, \dots, l_k \in \mathcal{V} \setminus \mathcal{V}_0} \sum_{i \in \mathcal{V}} \tilde{p}_i^{l_1 \cup \dots \cup l_k}, \quad (24)$$

thus one needs to search over the all k -tuples to determine the global optima. In the following, we will introduce a greedy suboptimum algorithm for the maximization problem and bound the level of suboptimality of the resulting solution

We first introduce the definition of a submodular function:

DEFINITION 1. Let \mathcal{V} be a finite set and f be a real-valued function defined on the collection of subsets of \mathcal{V} . The function f is called a submodular function if and only if

$$f(\mathcal{S}_1) + f(\mathcal{S}_2) \geq f(\mathcal{S}_1 \cup \mathcal{S}_2) + f(\mathcal{S}_1 \cap \mathcal{S}_2), \quad \text{for all } \mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}.$$

In the following, we will prove that the objective function of our maximization problem is indeed submodular.

LEMMA 2. *The objective function of the maximization problem in Eq. (24) is submodular.*

Since the objective function is submodular, we can use the greedy algorithm. We note that the greedy algorithm terminates in at most k steps, thus the complexity is $O(k(N - |\mathcal{V}_0|)^5)$. We can bound the performance characteristics of the above algorithm as follows:

COROLLARY 4. *For a given $k = K \geq 1$, denote the optimal and the greedy solutions of the maximization problem in Eq. (24) as \mathcal{S}^o and \mathcal{S}^g , respectively. Then, the following holds true:*

$$\left(1 - \frac{1}{e}\right) \sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}^o} \leq \left(1 - \left(\frac{K-1}{K}\right)^K\right) \sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}^o} \leq \sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}^g} \leq \sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}^o}$$

Kempe *et al.* used a similar greedy algorithm in the context of influence maximization (Kempe *et al.* (2003, 2005), Kleinberg (2007)). However, our update mechanism is not a special case of their framework, thus submodularity requires a proof which has been given in Lemma 2.

6.3. Strong Bottleneck Formation and Gluing Argument

While the discussions in Sections 6.1 and 6.2 provide explicit analytical results, they do not provide necessary intuition to understand where one should expect to place the stubborn agents for a given society \mathcal{G} and stubborn set \mathcal{V}_0 . In the following, we will try to fill that gap by analyzing absorption probabilities in more detail.

For a given society \mathcal{G} and the stubborn set of type zero agents \mathcal{V}_0 , let's assume that the set $\mathcal{S} \subset \mathcal{V} \setminus \mathcal{V}_0$ is the set of stubborn agents of type one. Then, by Eq. (24) the value of the objective function is equal to:

$$\sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}} \tag{25}$$

For each $i \in \mathcal{V} \setminus \mathcal{V}_0$, the absorption probability can be written as:

$$\tilde{p}_i^S = \frac{\mathbb{E}_i T_{\mathcal{V}_0} + \mathbb{E}_{\mathcal{V}_0} T_S - \mathbb{E}_i T_S}{K(\mathcal{S}, \mathcal{V}_0)}, \quad (26)$$

where $\mathbb{E}_i T_j$ is the expected first hitting time of the random walk on \mathcal{G} to node (or set) j when initialized at node i , and $K(i, j)$ is the mean commute time between nodes (sets) i and j ((Aldous and Fill 1994, Ch. 2)). Therefore, the term $\mathbb{E}_i T_{\mathcal{V}_0}$ is the expected hitting time to the set \mathcal{V}_0 . We can extend this particular term as:

$$\mathbb{E}_i T_{\mathcal{V}_0} = \mathbb{E}_i T_S + \mathbb{E}_S T_{\mathcal{V}_0} - \epsilon(i, \mathcal{S}, \mathcal{V}_0), \quad (27)$$

We note that $\epsilon(i, \mathcal{S}, \mathcal{V}_0) \geq 0$, since $\mathbb{E}_i T_S + \mathbb{E}_S T_{\mathcal{V}_0}$ corresponds to the case where the random walk first visits \mathcal{S} and then hits \mathcal{V}_0 , while $\mathbb{E}_i T_{\mathcal{V}_0}$ does not impose such a constraint. $\epsilon(i, \mathcal{S}, \mathcal{V}_0) \geq 0$ approximates the (expected) number of steps between i and \mathcal{V}_0 that do not go through \mathcal{S} . Moreover, $\epsilon(i, \mathcal{S}, \mathcal{V}_0) = 0$, if and only if all the paths from node i to the set \mathcal{V}_0 pass through \mathcal{S} , *i.e.*, \mathcal{S} is a bottleneck between i and \mathcal{V}_0 . Therefore, this particular term shows how strong of a bottleneck the set \mathcal{S} between node i and the set \mathcal{V}_0 .

Combining Eqs. (26) and (27):

$$\tilde{p}_i^S = \frac{\mathbb{E}_{\mathcal{V}_0} T_S + \mathbb{E}_S T_{\mathcal{V}_0} - \epsilon(i, \mathcal{S}, \mathcal{V}_0)}{K(\mathcal{S}, \mathcal{V}_0)} = 1 - \frac{\epsilon(i, \mathcal{S}, \mathcal{V}_0)}{K(\mathcal{S}, \mathcal{V}_0)}, \quad (28)$$

where the second equality follows from the definition of the mean commute time, *i.e.*, $K(\mathcal{S}, \mathcal{V}_0) = \mathbb{E}_{\mathcal{V}_0} T_S + \mathbb{E}_S T_{\mathcal{V}_0}$ ((Aldous and Fill 1994, Ch. 2)). Combining the above equation with Eq. (25), the value of the objective function (or the bias of the society towards the opinion "1") is:

$$\sum_{i \in \mathcal{V} \setminus \mathcal{V}_0} \tilde{p}_i^S = |N - \mathcal{V}_0| - \sum_{i \in \mathcal{V} \setminus \mathcal{V}_0} \frac{\epsilon(i, \mathcal{S}, \mathcal{V}_0)}{K(\mathcal{S}, \mathcal{V}_0)}. \quad (29)$$

Since we are searching for a set \mathcal{S} that maximizes the above summation, a suitable candidate should minimize the error terms $\epsilon(i, \mathcal{S}, \mathcal{V}_0)$, *i.e.*, should assume strong bottleneck properties between the society and the stubborn set \mathcal{V}_0 . In the cases where the underlying the graph \mathcal{G} and the location of the stubborn agents \mathcal{V}_0 do not permit such placement, a node with large commute time to the set \mathcal{V}_0 can be a suitable candidate.

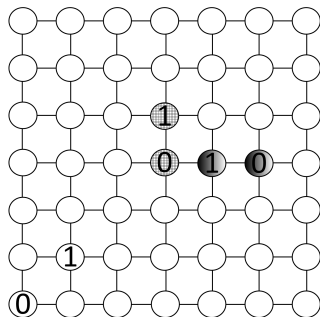


Figure 5 Pairs with the same background denote respective locations of type 0 and type 1 agents.

In the following, we will investigate several interesting graphs with different sizes of stubborn sets. In Figure 5, the underlying graph is a 7×7 square grid. Pairs with the same background denote respective locations of type 0 and type 1 agents in three different experiments. In each experiment, similar to our setup before, the location of the type 0 agent is fixed, and the optimal location of the type 1 agent is determined. For each of the cases, the optimal type 1 agent is close to the type 0 node and closer to the center of the society than the type 0 agent itself. Results coincide with our argument in Eq. (29), since the type 1 agent tries to form a bottleneck between the stubborn agent and the rest of the network in each of the cases. In Figure 6, we switch to a scenario where there are two stubborn agents of type 0, *i.e.*, $|\mathcal{V}_0| = 2$, and we need to place a single type 1 agent in the optimal way. When the stubborn set \mathcal{V}_0 is contained, *i.e.*, the elements of the set are close to each other, the behavior is similar to the case where there is a single stubborn agent of type 0. As the set \mathcal{V}_0 becomes distributed over the network, the type 1 agent preserves equal distance from each of the type 0 agents. In fact, this particular behavior also coincides with our "bottleneck" argument in Eq. (29). We note that if we glue the node in the set \mathcal{V}_0 together, *i.e.*, replace the set with single super node and rewire its connections accordingly, the outcome of our optimization problem remains unaffected. This argument follows from the fact that *gluing* vertices with same voltage does not change potentials which in return implies that the absorption probabilities stay the same (Levin et al. (2006)). In Figure 7, we introduce glued versions of the networks given in Figure 6. In the first three examples, *i.e.*, Figure 7 (a)-(c), type 1 agent is placed close to the type 0 agent's neighborhood and is closer to the center of the society than the type

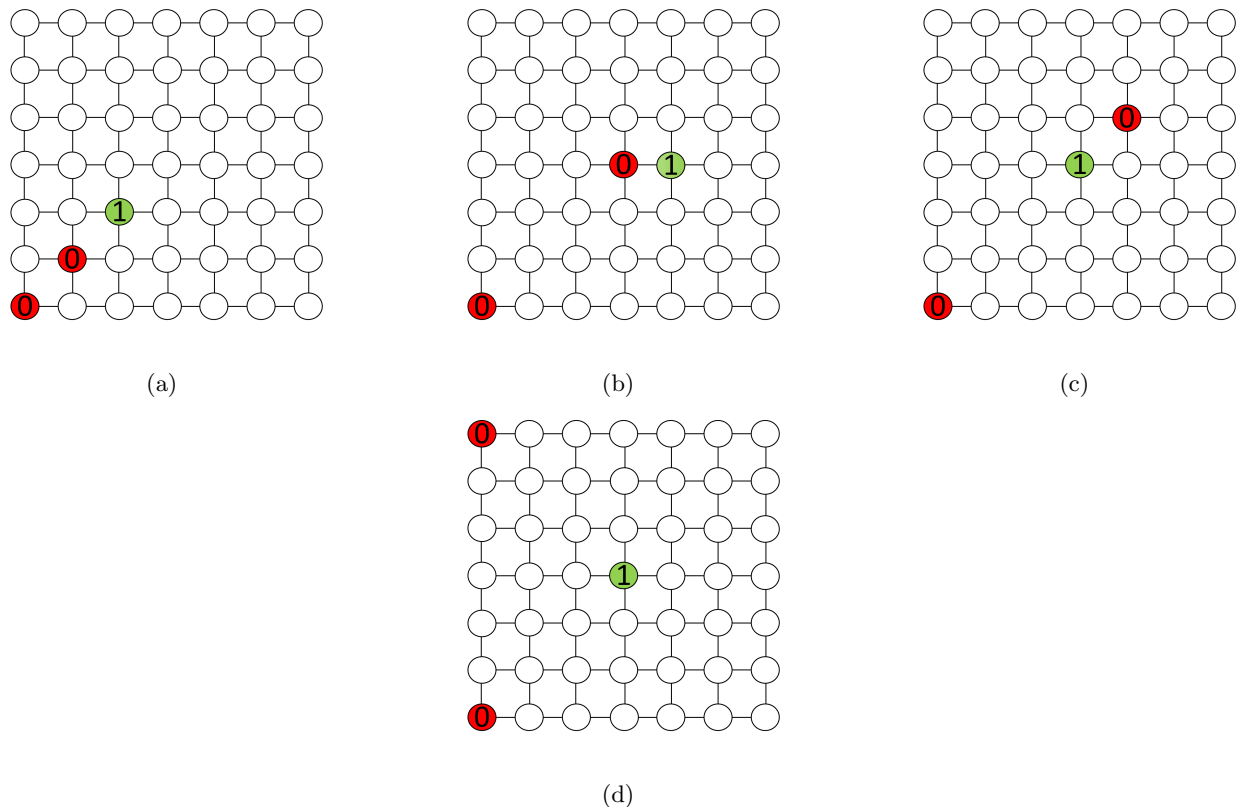


Figure 6 Different configurations of the set \mathcal{V}_0 and corresponding optimal type 1 agent locations.

0 agent itself. Thus, parallel to our argument in Eq. (29), type 1 agent tries to form a bottleneck between the society and the stubborn agent of type 0. We note that bottleneck formation is not as strong as the cases given in Figure 5, since type 0 agent’s neighbors are not confined. More interesting formation occurs in Figure 7 (d), where type 0 agent’s neighbors are sparsely distributed in the network. In this particular case, the optimal placement for the type 1 agent is closer to the center of the society, since favoring one of the distinct neighborhoods of the type 0 agent would result in weak bottleneck effect for the other neighborhood.

In Figure 8, we investigate a barbell graph with all to all connected side components. The sizes of the components are such that $|\mathcal{P}_1| = |\mathcal{P}_2| = |\mathcal{P}_3| > |\mathcal{L}_2| = |\mathcal{L}_1|$. There is a single type 0 stubborn agent, and optimal type 1 location is the boundary between the side component where the stubborn agent resides and the line component. Parallel to our argument in Eq. (29), type 1 agent forms a bottleneck between the significant portion of the network and the stubborn agent of type 0.

In Figure 9 (a)-(b), we once again focus on a barbell graph with all to all connected size compo-

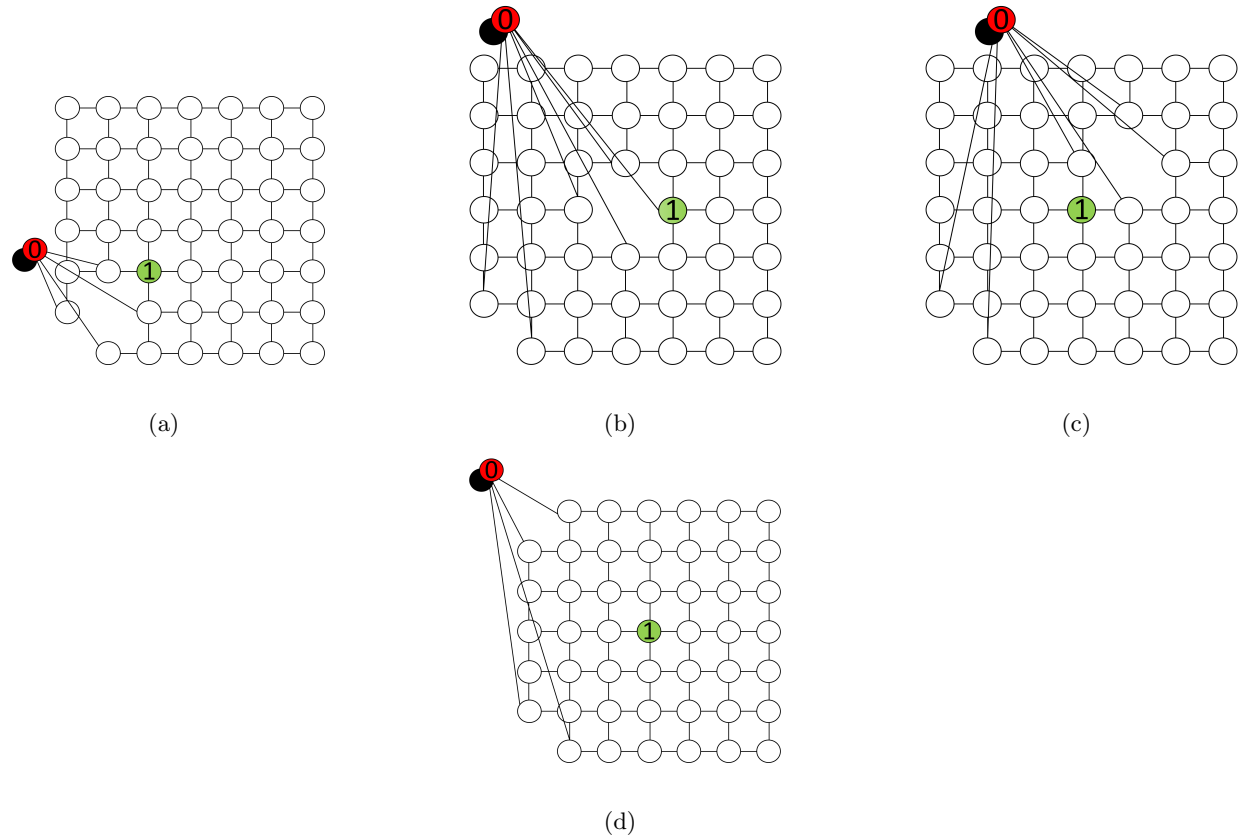


Figure 7 Different configurations of type 0 and corresponding optimal type 1 agent locations. Black nodes represent type 0 agents.

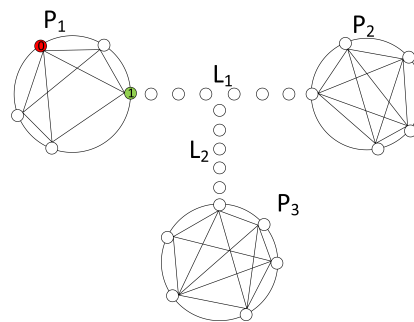


Figure 8 Barbell graph with all to all connected side components. $|P_1| = |P_2| = |P_3| > |L_1| = |L_2|$

ments and $|\mathcal{P}_1| = |\mathcal{P}_2| = |\mathcal{P}_3| > |\mathcal{L}_2| = |\mathcal{L}_1|$. In Figure 9 (a), there are two stubborn agents of type 0, located in \mathcal{P}_1 and \mathcal{P}_3 , respectively. The optimal location of the opposing agent is the intersection of the two line segments. If we glue the graph via the vertex where type 0 agents reside, the intersection of the two line segments indeed forms a bottleneck between the segment \mathcal{P}_2 and the stubborn agents on this new graph. Of note is that we highly rely on the assumption that the sizes of the

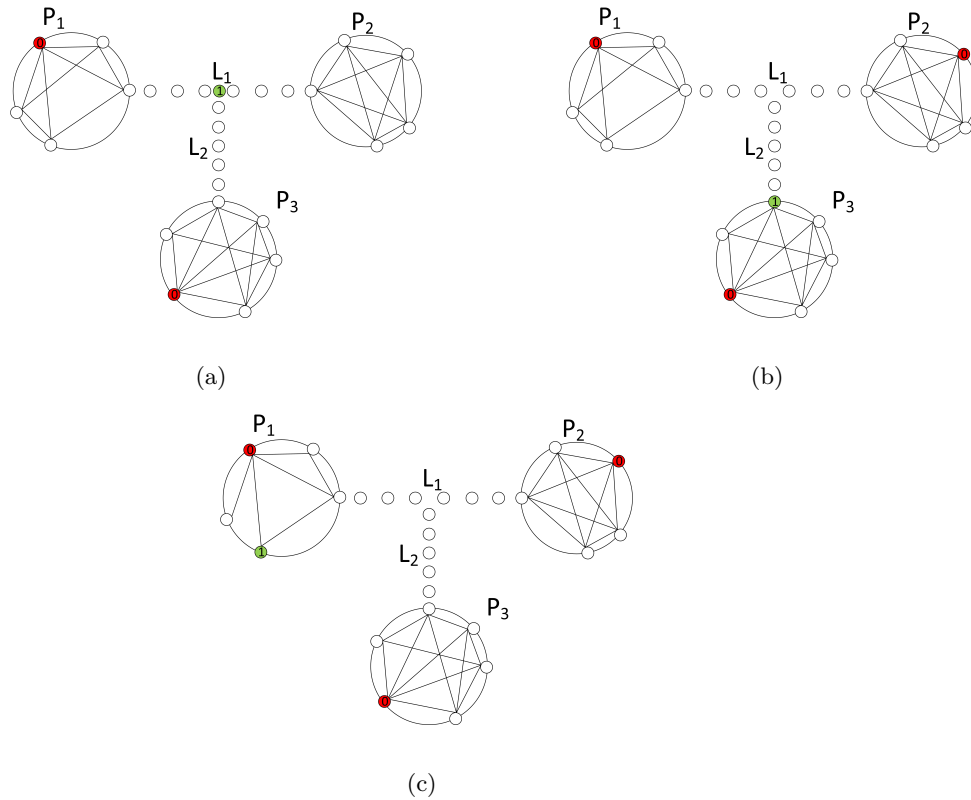


Figure 9 Different configurations of type 0 and corresponding optimal type 1 agent locations. Black nodes represent type 0 agents.

side components are on the same order, since results may change significantly if this assumption fail (see Figure 9 (c)). In Figure 9 (b), there are three stubborn agents which are distributed in \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{P}_3 . If we utilize the gluing approach, an influential location will be at the intersection of \mathcal{P}_3 and the line segment \mathcal{L}_2 , since stubborn agent of type 1 forms a bottleneck between the nodes on the line segment \mathcal{L}_2 and the stubborn node in \mathcal{P}_3 . We note that the intersections between \mathcal{P}_1 and \mathcal{L}_1 would not be the optimal choice since the line segment \mathcal{L}_1 has another stubborn agent of type 0 at its other end.

Figure 9 (c), we relax all to all connected assumption as well as the equality of the component sizes, *i.e.*,

$$|\mathcal{P}_1| \gg |\mathcal{P}_2| = |\mathcal{P}_3| > |\mathcal{L}_2| = |\mathcal{L}_1|. \quad (30)$$

In other words, the size of P_1 is significantly larger than the sizes of P_2 and P_3 . In this case, the

optimal location resides inside the sub-society \mathcal{P}_1 . If type 1 agent were to be replaced on the line segments, the nodes in \mathcal{P}_1 would reside in between the type 1 agent and the type 0, potentially closer to the type 0 node. Since the size of the sub society \mathcal{P}_1 is assumed to hold majority of the nodes in the network, type 1 agent would have to move inside the sub-society in order to increase its influence. Clearly, he has to pay a price, as he moves inside \mathcal{P}_1 , his influence on the subsets $\mathcal{P}_2, \mathcal{P}_3, \mathcal{L}_1, \mathcal{L}_2$ decreases. However, due to Eq. (30), the net change in his influence will be non-negative.

7. Conclusion

In this paper, we study a discrete opinion dynamics model in social networks with two distinct types of individuals, *i.e.*, stubborn and non-stubborn. Each individual can have two possible positions on a given subject, $\{0\}$ or $\{1\}$. Stubborn agents have fixed opinions while each non-stubborn agent periodically adopts the opinion of randomly chosen neighbor.

In this model, stubborn agents with opposing views prevent consensus in the society. However, under mild connectivity assumptions, we show that agent state vector converges in distribution. The stationary distribution of the agent state vector is independent of the initial opinions of the non-stubborn agents, thus the distribution of the agent state vector is completely determined by the underlying graph and the locations of the stubborn agents.

We next study the stationary distribution of the average opinion of the society $\bar{x}(t)$. Our analysis relies on an extension of the dual approach for the classical voter model, which involves coalescing random walks with absorbing states given by the stubborn agents. Our main results characterize the first two moments of the asymptotic average opinion. In particular, the expected value of the asymptotic opinion of agent i is given by the probability that the dual random walk on \mathcal{G} initiated at node i is absorbed by the stubborn set of type one. We characterize the fluctuations in the asymptotic average opinion of the society through the variance of $\bar{x}(t)$ in the limit.

Similar to the expected value of the average opinion, the variance of the asymptotic average opinion can be written in terms of the absorption probabilities of the dual process (coalescing

random walks) on the graph. We further show that the variance is given by a function of the unique solution of a system of harmonic equations. We introduce an upper bound for the variance of the asymptotic average opinion in terms of the graph structure as well as the locations of the stubborn agent. In particular, our upper bound is in terms of a novel node centrality measure, which captures the relative influence of that node with respect to the stubborn agent sets. In the bound, the centrality measure of a given node i is scaled by the variance of the asymptotic opinion of an agent i , therefore, the bound suggests that the variance of the asymptotic average opinion of the society may be higher when individuals with high relative influence are equally biased towards opinions 0 and 1. We discuss, by an example, that perturbations on the locations of the stubborn sets can transform a stable society into an unstable society. We introduce a convergence measure for the model, and characterize its scaling properties with respect to the graph size N in terms of the first hitting times of the dual process.

In the second part of the paper, we study the optimal stubborn agent placement problem, in which the goal is to choose the location of k stubborn agents of a given type (i.e., type one) to maximize the bias of the society. We assume that the underlying graph and the location of the stubborn agents of type zero are given. We first focus on the case $k = 1$ and provide a closed form expression for the optimal solution in terms of the fundamental matrix of the dual random walk. For $k > 1$, we show we show that the objective function is submodular, propose a greedy algorithm, and bound the level of suboptimality of the solution by $(1 - 1/e)$. We leave consideration of other metrics such as minimizing the variance of the asymptotic average opinion for future work.

Appendix. Proofs of Theorems, Lemmas, and Propositions

A. Proof of Theorem 1

We first note that we can characterize the behavior of the agent state vector $x(t)$ by a continuous time homogenous Markov Chain \mathcal{M} with the state space $\mathcal{S}_{\mathcal{M}}$. Each state of the chain is an N dimensional vector, where each dimension represents the value of a particular node in \mathcal{V} . Moreover,

the cardinality of the state space $\mathcal{S}_{\mathcal{M}}$ is $2^{N-|\mathcal{V}_0 \cup \mathcal{V}_1|}$, since stubborn agents do not update their opinions and the opinions of non-stubborn agents can only take two possible values, *i.e.*, 0 or 1.

We denote the transition rate matrix of \mathcal{M} as $Q_{\mathcal{M}}$. We define corresponding jump matrix as $P_{\mathcal{M}}$ (Norris 1998, Ch. 2):

$$[P_{\mathcal{M}}]_{ij} = \begin{cases} \frac{[Q_{\mathcal{M}}]_{ij}}{-[Q_{\mathcal{M}}]_i} & \text{if } j \neq i, [Q_{\mathcal{M}}]_i \neq 0, \\ 0 & j \neq i, [Q_{\mathcal{M}}]_i = 0, \\ 0 & j = i, [Q_{\mathcal{M}}]_i \neq 0, \\ 1 & j = i, [Q_{\mathcal{M}}]_i = 0, \end{cases} \quad (31)$$

where $[Q_{\mathcal{M}}]_i = -\sum_{j \in \mathcal{V}, j \neq i} [Q_{\mathcal{M}}]_{ij}$. In the following, we will first show that discrete time Markov Chain associated with $P_{\mathcal{M}}$, *i.e.*, the embedded chain, has a single recurrent class.

Without loss of generality, we assume that the chain has a non-empty set of transient states. Therefore, by potentially reordering the state space of \mathcal{M} , we can partition $P_{\mathcal{M}}$ as:

$$P_{\mathcal{M}} = \begin{bmatrix} P_1 & 0 \\ R & U \end{bmatrix}, \quad (32)$$

such that $\lim_{k \rightarrow \infty} U^k = 0$. The matrix U represents the transition probabilities between transient states. The set of indices corresponding to P_1 are the set of recurrent states. We will denote these indices as \mathcal{M}_1 . In the following, we will prove that the set of recurrent states forms a single class, *i.e.*, if $y, z \in \mathcal{M}_1$, then $y \leftrightarrow z$. Note that both y and z correspond to an N -dimensional state vector in the binary voter model. We denote these state vectors as x_y and x_z . Define the set $\mathcal{D} = \{m | [x_y]_m \neq [x_z]_m\}$, *i.e.*, the set of indices that are different in the configurations x_y and x_z . We first note that for each $m \in \mathcal{D}$, there exists a path from node m to at least one of the agents in the set \mathcal{V}_0 that does not go through \mathcal{V}_1 , and vice versa. We note that if this were not true, then at least one of x_y, x_z would be a transient state.

For a given $m \in \mathcal{D}$, determine a path from \mathcal{V}_1 to node $m \in \mathcal{V}$ which does not go through \mathcal{V}_0 . Then, consider the following chain of events: The first non-stubborn node along the path chooses a stubborn node in \mathcal{V}_1 and adopts its decision, the second non-stubborn node chooses the first non-stubborn node and adopts its decision, so on. It should be clear that this particular event has non-zero probability for state vectors x_y and x_z . Therefore, for a given $m \in \mathcal{D}$, this particular event

will transform x_y and x_z into $x_{y'}$ and $x_{z'}$ respectively, where $\mathcal{D}' = \{m | [x_{y'}]_m \neq [x_{z'}]_m\}$ is a *strict* subset of \mathcal{D} . If we apply the argument above for each $m \in \mathcal{D}$ sequentially, then x_y and x_z will be transformed into $x_{y^*} = x_{z^*}$.

We note that since y is a recurrent state and $y \rightarrow y^*$, then y^* is also a recurrent state, *i.e.*, $y \leftrightarrow y^*$. In the same way, one can show that $z \leftrightarrow z^*$. Finally, since $y^* = z^*$ and recurrence is a class property, $y \leftrightarrow z$. This completes the proof of the claim that discrete time Markov Chain associated with $P_{\mathcal{M}}$ has a single recurrent class.

We note that a state $y \in \mathcal{S}_{\mathcal{M}}$ in the continuous time Markov Chain \mathcal{M} is recurrent if and only if it is recurrent in the embedded chain given by Eq. (31) (Norris 1998, Ch. 3). Therefore, the continuous time chain \mathcal{M} has a single recurrent class. Moreover, since \mathcal{M} has a finite state space, *i.e.*, $|\mathcal{S}_{\mathcal{M}}| = 2^{N-|\mathcal{V}_0 \cup \mathcal{V}_1|}$ and finite arrival rates, *i.e.*, $\gamma = 1$, then the chain has a unique stationary distribution (Norris 1998, Ch. 3).

B. Proof of Proposition 1

Since $x_i(t)$ is a binary random variable for all $i \in \mathcal{V}$ and $t \geq 0$,

$$\lim_{t \rightarrow \infty} \mathbb{E}\{x_i(t)\} = \lim_{t \rightarrow \infty} \mathbb{P}(x_i(t) = 1), \quad (33)$$

where $\mathbb{P}(\cdot)$ is the probability of its argument. We note that, by our discussion in Section 2.3.2 and Eq. (3), we can track the source of the opinion of agent i at time $t \geq 0$ by the dual process $Y_i(t-s)$. Since all nodes are realized according to a rate 1 Poisson process and non-stubborn nodes adopt the opinion of randomly chosen neighbors, the dual process $Y_i(t-s)$ is a random walk with transition rates given in Eq. (6). Moreover, by choosing $s = 0$ in Eq. (3):

$$x_i(t) = x_{Y_i(t)}(0), \quad t \geq 0.$$

Since $Y_i(t)$ is a random walk on a finite graph \mathcal{G} , the connectivity of \mathcal{G} is such that there exists a path from each non-stubborn node to at least one stubborn node, and there exist two nonempty

absorbing classes \mathcal{V}_0 and \mathcal{V}_1 , then $\lim_{t \rightarrow \infty} Y_i(t) \in \{\mathcal{V}_0 \cup \mathcal{V}_1\}$ with probability 1 (see (Seneta 1981, Ch. 4.2)). By noting that $x_j(0) = 0$ if $j \in \mathcal{V}_0$, and $x_j(0) = 1$ if $j \in \mathcal{V}_1$,

$$\lim_{t \rightarrow \infty} \mathbb{P}(x_i(t) = 1) = \lim_{t \rightarrow \infty} \mathbb{P}(x_{Y_i(t)}(0) = 1) = \lim_{t \rightarrow \infty} \mathbb{P}(Y_i(t) \in \mathcal{V}_1). \quad (34)$$

By combining Eqs. (33) and (34),

$$\lim_{t \rightarrow \infty} \mathbb{E}\{x_i(t)\} = \lim_{t \rightarrow \infty} \mathbb{P}(Y_i(t) \in \mathcal{V}_1). \quad (35)$$

Moreover, since $x_i(t) \xrightarrow{D} x_i^*$ by Theorem 1, and $x_i(t)$ is a real valued random variable for each $i \in \mathcal{V}$ and $t \geq 0$, then:

$$\lim_{t \rightarrow \infty} \mathbb{E}\{f(x_i(t))\} = \mathbb{E}\{f(x_i^*)\},$$

for each bounded and continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$. If we construct $f^*(x)$ such that:

$$f^*(x) = \begin{cases} 0 & x < 0, \\ x & x \in [0, 1], \\ 1 & x > 1, \end{cases}$$

then $f^*(x)$ is bounded and continuous on its domain. Therefore,

$$\lim_{t \rightarrow \infty} \mathbb{E}\{f^*(x_i(t))\} = \mathbb{E}\{f^*(x_i^*)\}.$$

Moreover, since $x_i(t) \in \{0, 1\}$, $\mathbb{E}\{f^*(x_i(t))\} = \mathbb{E}\{x_i(t)\}$. Thus,

$$\lim_{t \rightarrow \infty} \mathbb{E}\{f^*(x_i(t))\} = \lim_{t \rightarrow \infty} \mathbb{E}\{x_i(t)\} = \mathbb{E}\{f^*(x_i^*)\} = \mathbb{E}\{x_i^*\}. \quad (36)$$

Combining Eq. (35) with Eq. (36), we obtain our result:

$$\mathbb{E}\{x_i^*\} = \lim_{t \rightarrow \infty} \mathbb{E}\{x_i(t)\} = \lim_{t \rightarrow \infty} \mathbb{P}(Y_i(t) \in \mathcal{V}_1).$$

C. Proof of Proposition 2

We first note that the equality:

$$\lim_{t \rightarrow \infty} \sigma(\bar{x}(t)) = \frac{1}{N^2} \sum_{i,j \in \mathcal{V}} \left(\sum_{m,n \in \mathcal{V}_1} \lim_{t \rightarrow \infty} \mathbb{P}(Z'(t) = (m,n) | Z'(0) = (i,j)) - p_i p_j \right),$$

follows from Lemma 1 and Eq. (8). Therefore, we only need to show the following equality:

$$\sigma(x^*) = \lim_{t \rightarrow \infty} \sigma(\bar{x}(t)).$$

Since $\bar{x}(t) \xrightarrow{D} x^*$ by Corollary 1, and $\bar{x}(t)$ is a real valued random variable for $t \geq 0$, then:

$$\lim_{t \rightarrow \infty} \mathbb{E} \{f(\bar{x}(t))\} = \mathbb{E} \{f(\bar{x}^*)\},$$

for each bounded and continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$. If we construct $f^*(x)$ such that:

$$f^*(x) = \begin{cases} 0 & x < 0, \\ x^2 & x \in [0, 1], \\ 1 & x > 1, \end{cases}$$

then $f^*(x)$ is bounded and continuous on its domain. Therefore,

$$\lim_{t \rightarrow \infty} \mathbb{E} \{f^*(\bar{x}(t))\} = \mathbb{E} \{f^*(\bar{x}^*)\}.$$

Moreover, since $\bar{x}(t) \in \{0, 1\}$, $\mathbb{E} \{f^*(\bar{x}(t))\} = \mathbb{E} \{(\bar{x}(t))^2\}$. Thus,

$$\lim_{t \rightarrow \infty} \mathbb{E} \{f^*(\bar{x}(t))\} = \lim_{t \rightarrow \infty} \mathbb{E} \{(\bar{x}(t))^2\} = \mathbb{E} \{f^*(\bar{x}^*)\} = \mathbb{E} \{(\bar{x}^*)^2\}. \quad (37)$$

By the same way, one can also show that $\lim_{t \rightarrow \infty} \mathbb{E} \{\bar{x}(t)\} = \mathbb{E} \{\bar{x}^*\}$. Therefore,

$$\sigma(x^*) = \lim_{t \rightarrow \infty} \sigma(\bar{x}(t)).$$

D. Proof of Proposition 3

We first define the event I_{ij}^l as follows: For $l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$, I_{ij}^l is the event that the random walk $Z'(t)$ on \mathcal{G}' , initialized at the node (i, j) , hits the node (l, l) before being absorbed by the absorbing states or before hitting any other node of the form (l', l') , $l' \in \mathcal{V} \setminus l$. In other words, I_{ij}^l is the event that the coalescing random walks on \mathcal{G} , initialized at nodes i and j , coalesce at node $l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$ before being absorbed. Therefore,

$$\begin{aligned} \sum_{m, n \in \mathcal{V}_1} \lim_{t \rightarrow \infty} \mathbb{P}(Z'(t) = (m, n) | Z'(0) = (i, j)) &= \sum_{m, n \in \mathcal{V}_1} \sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}(I_{ij}^l) \lim_{t \rightarrow \infty} \mathbb{P}(Z'(t) = (m, n) | Z'(0) = (l, l)) \\ &\quad + \mathbb{P}(A_{ij}) \\ &= \sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}(I_{ij}^l) p_l + \mathbb{P}(A_{ij}), \end{aligned}$$

where A_{ij} is the event that the walk $Z'(t)$ is absorbed by the absorbing set before hitting (l, l) , $l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$. Moreover,

$$p_i p_j = \sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}(I_{ij}^l) p_l^2 + \mathbb{P}(A_{ij}),$$

since this particular case corresponds to two independent random walks on the graph \mathcal{G} . Therefore, by Proposition 2, the variance of the asymptotic average opinion is equal to:

$$\sigma(\bar{x}^*) = \frac{1}{N^2} \sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} (p_l - p_l^2) \sum_{i, j \in \mathcal{V}} \mathbb{P}(I_{ij}^l), \quad (38)$$

Finally, for a given l , $\mathbb{P}(I_{ij}^l)$ is a non-negative harmonic function with the following conditions (Aldous and Fill (1994)):

$$\mathbb{P}(I_{ij}^l) = \begin{cases} 0 & \text{if } i = j \neq l, \text{ or } , i \in \{\mathcal{V}_1 \cup \mathcal{V}_0\}, \text{ or } , j \in \{\mathcal{V}_1 \cup \mathcal{V}_0\}, \\ 1 & \text{if } i = j = l, \\ \frac{1}{2|\mathcal{N}_i|} \sum_{i' \in \mathcal{N}_i} \mathbb{P}(I_{i'j}^l) + \frac{1}{2|\mathcal{N}_j|} \sum_{j' \in \mathcal{N}_j} \mathbb{P}(I_{ij'}^l) & \text{otherwise.} \end{cases}$$

A unique solution for the above harmonic system exists (Aldous and Fill (1994)).

E. Proof of Theorem 2

In the following, we will bound the term $\mathbb{P}(I_{ij}^l)$ in Eq. (12) for each $i, j \in \mathcal{V}$ and $l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$. We first note that I_{ij}^l is the event that the coalescing random walks on \mathcal{G} , initialized at nodes i and j , coalesce at node $l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$ before being absorbed. At this point, for a given (i, j, l) , we modify the coalescing random walks on \mathcal{V} as follows: The coalescing random walks initialized at i and j can only coalesce at l , *i.e.*, if the individual random walks meet at a given node $l' \notin \{\mathcal{V}_0 \cup \mathcal{V}_1 \cup l\}$, they will not coalesce. We further assume that node l is an absorbing state, *i.e.*, if either i or j hits node l , it will be absorbed by that particular node. At this point, we observe that the probability that the modified process coalescing at node l is greater than equal to $\mathbb{P}(I_{ij}^l)$. Finally, the coalescing probability in the modified process is simply $\tilde{p}_i^l \tilde{p}_j^l$. Using the fact that $\tilde{p}_i^l \tilde{p}_j^l \geq \mathbb{P}(I_{ij}^l)$ for all $i, j \in \mathcal{V}, l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$, our result follows.

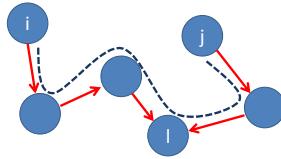


Figure 10 A sample path for coalescing walks initiated at node i and j respectively. They collide at node l . The superimposed path (dashed line) is a walk from node i to node j .

F. Proof of Proposition 4

We assume that \mathcal{G} is a d regular graph with symmetric edges. Using Eq. (12), one can show that:

$$\sigma(\bar{x}^*) \leq \frac{1}{4N^2} \sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \sum_{i, j \in \mathcal{V}} \mathbb{P}(I_{ij}^l) = \frac{1}{4N^2} \sum_{i, j \in \mathcal{V}} \sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}(I_{ij}^l), \quad (39)$$

where the first inequality follows from the fact that $(p_l - p_l^2) \leq 1/4$, for all $l \in \mathcal{V}$. At this point, we would like to note that for a given (i, j) pair, $\sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}(I_{ij}^l)$ is the sum probability that the coalescing random walks initiated at nodes i and j respectively, coalesce in $\mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}$ before being absorbed by $\mathcal{V}_1 \cup \mathcal{V}_0$. We consider the trajectory of random walks started from i and j until they meet. If we superimpose these two walks on top of each other, we will form a walk from node i to j (and vice versa). An illustration is given in Figure 10.

Let $\mathcal{W}_{ij} = \{i, t_1, t_2, \dots, j\}$ be a trajectory between nodes i and j which does not visit the set $\mathcal{V}_0 \cup \mathcal{V}_1$. If we consider the random walk on the graph \mathcal{G} initiated at node i , \mathcal{W}_{ij} will have the probability $d^{-|\mathcal{W}_{ij}|+1}$. We denote this probability as $\mathbb{P}(\mathcal{W}_{ij})$. If we consider the coalescing walks initiated on i and j on that particular trajectory, then, the probability that these two walks meet on $1 \leq t \leq |\mathcal{W}_{ij}|$ -th element of the trajectory \mathcal{W}_{ij} is equal to:

$$\mathbb{P}(\text{the coalescing walks meeting at the } t\text{-th element of the trajectory } \mathcal{W}_{ij}) = (2d)^{-|\mathcal{W}_{ij}|+1} \binom{|\mathcal{W}_{ij}|-1}{t-1}, \quad (40)$$

where $\binom{\cdot}{\cdot}$ is the combination operator. 2 factor in front is due to the fact that the coalescing walks initiated at nodes i and j follows transition rates given in Eq. (11). If we sum both sides of Eq. (40) over $t \in \{1, \dots, |\mathcal{W}_{ij}|\}$,

$$\begin{aligned}
\sum_{t=0}^{|\mathcal{W}_{ij}|} \mathbb{P}(i, j \text{ meet at the } t\text{-th element of the trajectory } \mathcal{W}_{ij}) &= (2d)^{-|\mathcal{W}_{ij}|+1} \sum_{t=1}^{|\mathcal{W}_{ij}|} \binom{|\mathcal{W}_{ij}|-1}{t-1} \\
&= (2d)^{-|\mathcal{W}_{ij}|+1} (2)^{|\mathcal{W}_{ij}|-1} = d^{-|\mathcal{W}_{ij}|+1} = \mathbb{P}(\mathcal{W}_{ij}), \tag{41}
\end{aligned}$$

where the second equality follows from summation of binomial coefficient series. Therefore, for a given trajectory \mathcal{W}_{ij} , the sum of the probabilities (over the elements of the trajectory) that walks initiated at node i and j meet on a given element of the trajectory is equal to the probability of the trajectory \mathcal{W}_{ij} for the random walk initiated at node i .

To determine the overall coalescing probability of the walks initiated at nodes i and j , we need to sum over the all trajectories \mathcal{W}_{ij} none of which hits $\mathcal{V}_1 \cup \mathcal{V}_0$. Denoting, $\{T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}^i$ as the event that the random walk initiated at node i hits state j before hitting $\{\mathcal{V}_0 \cup \mathcal{V}_1\}$, $\mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1})$ as its probability, $\{T^{k+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}^j$ as the event that random initiated at node j returns itself k times before hitting the set $\mathcal{V}_0 \cup \mathcal{V}_1$, and $\mathbb{P}_j(T_j^{k+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1})$ as its probability:

$$\sum_{l \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}(I_{ij}^l) \leq \sum_{\mathcal{W}_{ij}} \sum_{t=0}^{|\mathcal{W}_{ij}|} \mathbb{P}(i, j \text{ meet at the } t\text{-th element of the trajectory } \mathcal{W}_{ij}) \tag{42}$$

$$= \sum_{\mathcal{W}_{ij}} \mathbb{P}(\mathcal{W}_{ij}) \tag{43}$$

$$= \mathbb{P}(\{T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}^i) + \sum_{k=1}^{\infty} \mathbb{P}(\{T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}^i \cap \{T_j^{k+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}^j) \tag{44}$$

$$= \mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1}) + \sum_{k=1}^{\infty} \mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1}) \mathbb{P}_j(T_j^{k+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1}) \tag{45}$$

$$= \mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1}) \sum_{k=0}^{\infty} (\mathbb{P}_j(T_j^{1+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1}))^k \tag{46}$$

$$= \frac{\mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1})}{1 - \mathbb{P}_j(T_j^{1+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1})}, \tag{47}$$

where the inequality in Eq. (42) is due to the fact that some of the absorption paths have been counted more than once, Eq. (43) follows from Eq. (41), Eq. (44) is due to conditioning on the events that the random walk returns node j k -times before before hitting $\{\mathcal{V}_0 \cup \mathcal{V}_1\}$, and Eqs. (45)-(46) follow from the Markovian nature of the process, and Eq. (47) can be obtained by geometric series sum formula.

If we combine Eqs. (47) and (39), we obtain the following:

$$\begin{aligned} \sigma(\bar{x}^*) &\leq \frac{1}{4N^2} \sum_{i,j \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \frac{\mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1})}{1 - \mathbb{P}_j(T_j^{1+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1})} \\ &= \frac{1}{4N^2} \sum_{j \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \left\{ (1 - \mathbb{P}_j(T_j^{1+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1}))^{-1} \sum_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1}) \right\}. \end{aligned}$$

We first note that the summation term $\sum_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{P}_i(T_j < T_{\mathcal{V}_0 \cup \mathcal{V}_1})$ is equal to the relative influence of node j , *i.e.*, r^j .

$$(1 - \mathbb{P}_j(T_j^{1+} < T_{\mathcal{V}_0 \cup \mathcal{V}_1}))^{-1} = (\mathbb{P}_j(T_{\mathcal{V}_0 \cup \mathcal{V}_1} < T_j^{1+}))^{-1} = d R(j \leftrightarrow (\mathcal{V}_0 \cup \mathcal{V}_1)),$$

where $R(\cdot)$ is the effective resistance between its arguments assuming that each edge in \mathcal{E} has a unit conductance (Levin et al. 2006, Ch. 2). Therefore, we obtain the following upper bound on the variance:

$$\sigma(\bar{x}^*) \leq \frac{d}{4N^2} \sum_{j \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} R(j \leftrightarrow (\mathcal{V}_0 \cup \mathcal{V}_1)) r^j.$$

G. Proof of Lemma 5

We first introduce a lemma by Aldous on the exponentiality of the tails of the hitting times on Markov chains.

LEMMA 3. (Aldous and Fill 1994, Ch. 2.4.3) For a given finite-state irreducible Markov Chain \mathcal{M} with state space \mathcal{S} , define $T_{\mathcal{A}}^i$ as the first hitting time on a subset $\mathcal{A} \subset \mathcal{S}$ when the chain is initialized at node i . Denoting $\mathbb{E}_i T_{\mathcal{A}}$ as the expected value of $T_{\mathcal{A}}^i$ and $m = \max_{i \in \mathcal{S}} \mathbb{E}_i T_{\mathcal{A}}$,

$$\max_{i \in \mathcal{S}} \mathbb{P}(T_{\mathcal{A}}^i > t) \leq \exp\left(-\lfloor \frac{t}{em} \rfloor\right),$$

where \exp is the exponential function, e is the exponential coefficient and $\lfloor \cdot \rfloor$ is the floor of its argument.

We first note that the dual process $Y_{\mathcal{V}}(t)$ consists of N individual random walks. Therefore:

$$T_{\mathcal{V}_0 \cup \mathcal{V}_1} \leq \max_{i \in \mathcal{V}} \{T_{\mathcal{V}_0 \cup \mathcal{V}_1}^i\}, \quad (48)$$

where $T_{\mathcal{V}_0 \cup \mathcal{V}_1}^i = \arg \min_{t \geq 0} \{Y_i(t) \in \{\mathcal{V}_0 \cup \mathcal{V}_1\}\}$, for all $i \in \mathcal{V}$. We note that $T_{\mathcal{V}_0 \cup \mathcal{V}_1}^i$ is the first hitting time of the random walk $Z(t)$ in Eq. (6) to the set $\{\mathcal{V}_0 \cup \mathcal{V}_1\}$, when initialized at node i .

In the light of Lemma 3 and the above discussion, the convergence time measure $\mathbb{E}\{T_{\mathcal{V}_0 \cup \mathcal{V}_1}\}$ can be bounded as:

$$\begin{aligned} \mathbb{E}\{T_{\mathcal{V}_0 \cup \mathcal{V}_1}\} &= \int_{t=0}^{\infty} \mathbb{P}(T_{\mathcal{V}_0 \cup \mathcal{V}_1} > t) dt \\ &\leq \int_{t=0}^{\infty} \mathbb{P}\left(\max_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} T_{\mathcal{V}_0 \cup \mathcal{V}_1}^i > t\right) dt \\ &\leq \int_{t=0}^{\infty} \min\left(1, (N - |\mathcal{V}_0 \cup \mathcal{V}_1|) \exp\left(\frac{-t}{e \max_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{E}_i T_{\mathcal{V}_0 \cup \mathcal{V}_1}}\right)\right) dt \\ &= e(2 + (N - |\mathcal{V}_0 \cup \mathcal{V}_1|)) \max_{i \in \mathcal{V} \setminus \{\mathcal{V}_0 \cup \mathcal{V}_1\}} \mathbb{E}_i T_{\mathcal{V}_0 \cup \mathcal{V}_1}, \end{aligned}$$

where the first equality follows from the fact that $T_{\mathcal{V}_0 \cup \mathcal{V}_1}$ is non-negative, the first inequality is due to Eq. (48), and the last equality follows from fact that (Aldous and Fill 1994, Ch. 5.3.2)

$$\int_{t=0}^{\infty} \min(1, Ae^{-at}) dt = \frac{1 + \log A}{a}, \quad A \geq 1.$$

H. Proof of Lemma 2

Let \mathcal{S}_1 and \mathcal{S}_2 be two non-empty subsets of $\mathcal{V} \setminus \mathcal{V}_0$. Denoting $\mathbb{P}_i(T_u < T_v)$ as the probability that the random walk initiated at node i hits state u before hitting v ,

$$\mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0}) = \mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_1} < T_{\mathcal{S}_2}) + \mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_1} = T_{\mathcal{S}_2}) + \mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_1} > T_{\mathcal{S}_2}), \quad (49)$$

where Eq. (49) is due to the partitioning theorem Jacod and Protter (2000). By the same argument, the following also holds true:

$$\mathbb{P}_i(T_{\mathcal{S}_2} < T_{\mathcal{V}_0}) = \mathbb{P}_i(T_{\mathcal{S}_2} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_2} < T_{\mathcal{S}_1}) + \mathbb{P}_i(T_{\mathcal{S}_2} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_2} = T_{\mathcal{S}_1}) + \mathbb{P}_i(T_{\mathcal{S}_2} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_2} > T_{\mathcal{S}_1}). \quad (50)$$

Moreover, in the case that \mathcal{S}_1 and \mathcal{S}_2 are merged to form a single class,

$$\mathbb{P}_i(T_{\mathcal{S}_1 \cup \mathcal{S}_2} < T_{\mathcal{V}_0}) = \mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_1} < T_{\mathcal{S}_2}) + \mathbb{P}_i(T_{\mathcal{S}_2} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_2} < T_{\mathcal{S}_1}) \quad (51)$$

$$+ \mathbb{P}_i(T_{\mathcal{S}_1 \cap \mathcal{S}_2} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_1 \cap \mathcal{S}_2} < T_{\mathcal{S}_1 \setminus \mathcal{S}_2} \cap T_{\mathcal{S}_1 \cap \mathcal{S}_2} < T_{\mathcal{S}_2 \setminus \mathcal{S}_1}), \quad (52)$$

where the first term in Eq. (51) is the probability that the random walk hits to the set \mathcal{S}_1 before hitting $\{\mathcal{V}_0 \cup \mathcal{S}_2\}$, the second term is the probability that the random walk hits to the set \mathcal{S}_2 before hitting $\{\mathcal{V}_0 \cup \mathcal{S}_1\}$, and the term in Eq. (52) is the the probability that the random walk hits to the set $\mathcal{S}_1 \cap \mathcal{S}_2$ before hitting $\{\mathcal{V}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \setminus (\mathcal{S}_1 \cap \mathcal{S}_2)\}$. Finally,

$$\mathbb{P}_i(T_{\mathcal{S}_1 \cap \mathcal{S}_2} < T_{\mathcal{V}_0}) \leq \mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0} \cap T_{\mathcal{S}_2} = T_{\mathcal{S}_1}), \quad (53)$$

where the inequality follows from the fact that $T_{\mathcal{S}_2} = T_{\mathcal{S}_1}$ is the set of events where the hitting times to sets \mathcal{S}_1 and \mathcal{S}_2 are equal. We note that this is a necessary condition for the random walk hitting the intersection.

Combining Eqs. (49)-(53), we can show that:

$$\mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0}) + \mathbb{P}_i(T_{\mathcal{S}_2} < T_{\mathcal{V}_0}) \geq \mathbb{P}_i(T_{\mathcal{S}_1 \cup \mathcal{S}_2} < T_{\mathcal{V}_0}) + \mathbb{P}_i(T_{\mathcal{S}_1 \cap \mathcal{S}_2} < T_{\mathcal{V}_0}). \quad (54)$$

By noting that $\mathbb{P}_i(T_{\mathcal{S}_1} < T_{\mathcal{V}_0}) = \tilde{p}_i^{\mathcal{S}_1}$ by definition and summing over all $i \in \mathcal{V}$, we conclude our proof:

$$\sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}_1} + \sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}_2} \geq \sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}_1 \cup \mathcal{S}_2} + \sum_{i \in \mathcal{V}} \tilde{p}_i^{\mathcal{S}_1 \cap \mathcal{S}_2}.$$

Endnotes

1. Throughout the paper, we will use the terms agent, individual, and node interchangeably. Similarly, the terms network and graph will be used interchangeably.

2. The voter model is a generalization where agent opinions take values in a finite set \mathcal{T} . The subsequent analysis can be extended to this case with straightforward modifications.

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