Efficiency and Braess Paradox under Pricing

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Resource and Traffic Management in Communication Networks

- Flow control and routing essential components of traffic management.
- Traditional Network Optimization: Focus on a central objective, devise synchronous/asynchronous, centralized/distributed algorithms.
  - Assumes all users are homogeneous with no self interest
- Today's Large-scale Networks (eg. Internet):
  - Decentralized operation
  - Highly heterogeneous nature of users
  - Interconnection of privately owned networks

Emerging Paradigm for Distributed Control

- Analysis of resource allocation in the presence of decentralized information, selfish users/administrative domains, and profit-maximizing service providers.
- Instead of a central control objective, model as a multi-agent decision problem.
  - Some control functions delegated to agents with independent objectives.
  - Suggests using game theory and economic market mechanisms.

Recent Literature

- Flow (congestion) control by maximizing aggregate source utility over transmission rates
  - "Kelly mechanism": Decentralized incentive compatible resource allocation [Kelly 97], [Kelly, Maulloo, Tan 98]
  - Primal/Dual methods, stability, relations to current congestion control mechanisms [Low, Lapsley 02], [Liu, Basar, Srikant 03]
- Selfish (user-directed) routing
  - Transportation net. [Wardrop 52],[Beckmann 56],[Patriksson 94]
  - Communication networks [Orda, Rom, Shimkin 93], [Korilis, Lazar, Orda 97], [Roughgarden, Tardos 00]
- Efficiency
  - "Price of Anarchy": Ratio of performance of selfish to performance of social [Koutsoupias, Papadimitriou 99], [Roughgarden, Tardos 00], [Correa, Schulz, Stier Moses 03], [Johari, Tsitsiklis 03]

Previous Work

- Existing literature focuses on:
  - resource allocation among competing heterogeneous users
  - social welfare (aggregate utility) maximization
- Pricing used as a means of regulating selfish user behavior and achieving social optimum in a distributed manner .
- Commercial networks operated by for-profit service providers.
  - Pricing used to make profits or provide service differentiation among users.
  - Combined study of pricing and resource allocation essential in the design of networks.
- With a few exceptions ([He, Walrand 03], [Mitra et al. 01], [Basar, Srikant 02]), this game theoretic interaction neglected.
- In [Acemoglu, Ozdaglar 04], we studied pricing with combined flow control/routing for parallel link networks.

This Talk

- Consider selfish flow choice and routing in a general topology network where resources are owned by for-profit entities (focus on a single service provider).
- Each user pays a price proportional to the amount of bandwidth she uses (usage-based, linear pricing).
- Goal: Develop a framework and study the implications of pricing on various performance results.
- Two parts:
  - Equilibrium and efficiency of combined flow control/routing
  - Braess' paradox under pricing

Model for Decentralized System

- Directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, m origin destination pairs
- For each link e, a latency function $l^e : [0, C^e] \mapsto [0, \infty)$, where C^e denotes the capacity of link e
  - specifies the delay on the link given its congestion.
- For each source destination pair k, $\mathcal{J}_k$ set of users, $\mathcal{P}_k$ set of paths
- For each user $j \in J_k$, a utility function $u_{k,j} : [0, \infty) \mapsto [0, \infty)$
  - measure of benefits from data transmission.
- Depending on application service requirements, utility takes different forms [Shenker 95]:
  - Inelastic applic: real time voice, video (step utility function)
  - Elastic applic: e-mail (increasing concave utility function)
- Single service provider: charges q^p per unit bw on path p.

User Equilibrium - Wardrop Equilibrium

- Let $f^p_{k,j}$: flow of user $j \in \mathcal{J}_k$ on path p.

$$\Gamma_{k,j} = \sum_{p \in \mathcal{P}_k} f^p_{k,j}, \qquad \text{“ flow rate of user } j\text{”}$$

$$f^p = \sum_{j \in \mathcal{J}_k} f^p_{k,j}, \qquad \text{“flow on path } p\text{”}$$

$$f^{\mathcal{E}} = [f^1, \ldots, f^{|\mathcal{E}|}], \qquad \text{“vector of link loads”}$$

- Let payoff function v_j of user j be defined by

$$v_j(f_{k,j}; f^{\mathcal{E}}, q) = u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \mathcal{P}_k} \sum_{e \in p} l^e(f^e) \Bigg) f^p_{k,j} - \sum_{p \in \mathcal{P}_k} q^p f^p_{k,j}.$$

- Definition: For a given price $q \geq 0$, f^* is a Wardrop Equilibrium if

$$f^*_{k,j} \in \arg \max_{f_{k,j} \geq 0} v_j(f_{k,j}; f^{\mathcal{E}}, q), \quad \forall\ j \in J_k, \ \forall\ k,$$

$$f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{p | e \in p, p \in \mathcal{P}_k} (f^*)^p_{k,j}, \qquad \forall\ e \in \mathcal{E}.$$

- Implicit Assumption: Users small relative to the network

Single Service Provider

- The SP sets prices per unit bandwidth for each path to maximize profits.
- Monopoly Problem:

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_p q^p f^p(q) \\ \text{subject to} \quad & q \geq 0, \end{aligned}$$

  where $f^p(q)$ is the flow on path p at the WE given price vector q.
- This problem has an optimal solution q^*.
- We will refer to q^* as the monopoly equilibrium price and $(q^*, f(q^*))$ [or (q^*, f^*)] as the monopoly equilibrium (ME).

Elastic Traffic

- The utility function $u_{k,j}$ is concave and nondecreasing.
- ME price: Let (q, f) be an ME and let $\bar{\mathcal{J}}_k = \{j \mid j \in \mathcal{J}_k,\ \Gamma_{k,j} > 0\}$. Then,

$$q^p = \quad \sum_{e \in p} (l^e)'(f^e) f^e \Bigg) + \frac{\sum_{p \in \bar{\mathcal{P}}_k} f^p}{-\sum_{j \in \bar{\mathcal{J}}_k} \frac{1}{u''_{k,j}(\Gamma_{k,j})}}.$$

- Social Problem

$$\text{maximize} \quad \sum_{j \in \mathcal{J}_k} u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \mathcal{P}_k} \quad \sum_{e \in p} l^e(f^e) \quad f^p$$

- Equivalent characterization of social opt: (assuming l^i is convex)

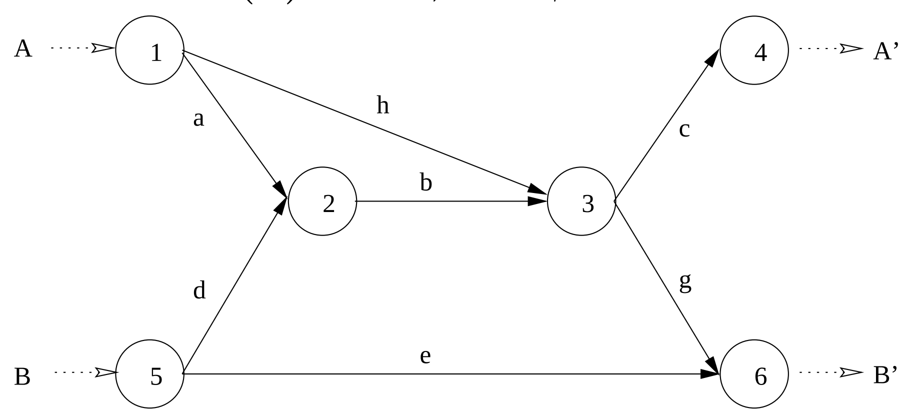
$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p} l^e(f^e) - \sum_{e \in p} (l^e)'(f^e) f^e \quad \begin{cases} \leq 0, & \text{if } f^p_{k,j} = 0, \\ = 0, & \text{if } f^p_{k,j} > 0. \end{cases}$$

- ME price= Marginal congestion cost + Monopoly markup
- For linear utility functions, path flows and flow rates of the ME and the social optimum are the same.

Performance of Monopoly Pricing

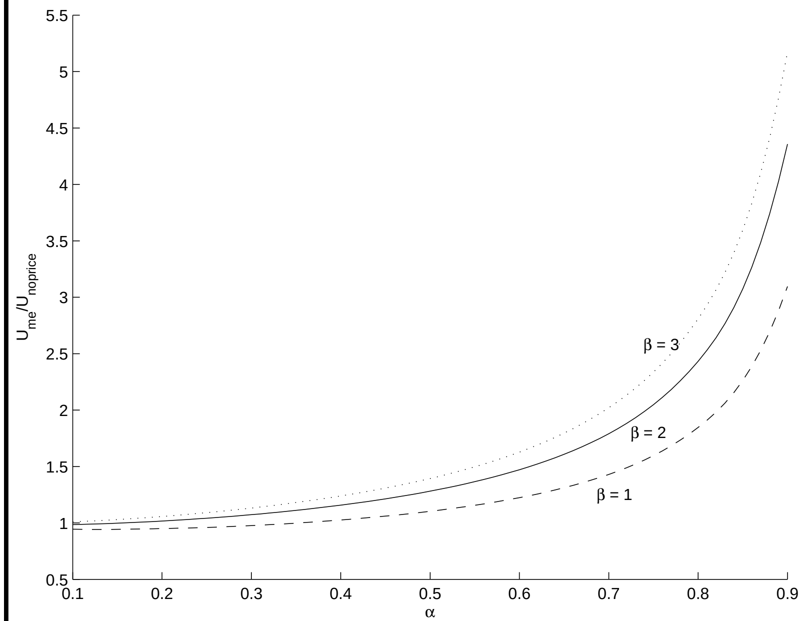
- Compare performance of monopoly pricing, WE(0), and social opt.
- Example: Consider the network below. Assume

$$u_A(x) = u_B(x) = 200x^{\alpha}, \qquad 0 < \alpha \leq 1,$$
$$l(x) = x^{\beta}, \qquad \beta \geq 1.$$

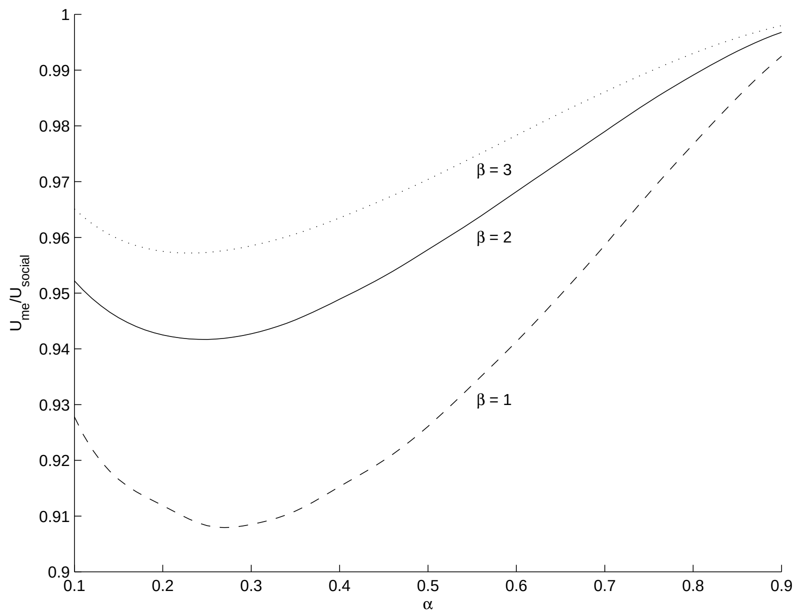


- U_{me}, U_{we}, U_{soc}: total system utility at the ME, WE(0), and social optimum.

Exploiting Convexity of Latency Functions



U_{me}/U_{we}

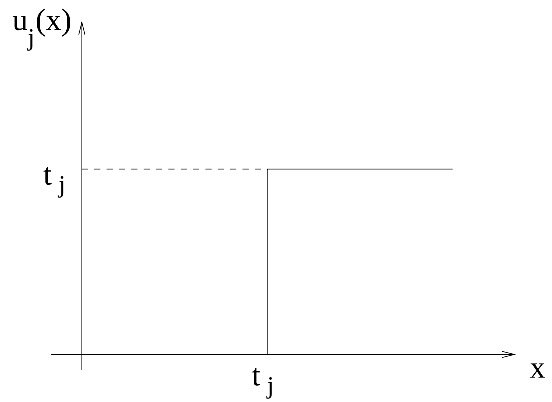


U_{me}/U_{soc}

- The more convex the latency function is, the better performance we have under monopoly pricing.
- Concavity in utility introduces distortion wrt to the social optimum.

Inelastic Traffic - Routing (with participation control)

- The utility function u_j is a step function.



- With this utility function, decisions of user j will be binary: either send t_j units of traffic or do not send anything.
  - Routing with participation control
- Reasonable model of routing in the presence of service providers
  - Otherwise, the monopolist will set the prices equal to ∞.
- Includes implicit admission control.

Analysis of Inelastic Traffic

- The u_j are no longer concave or continuous, therefore calculus-based analysis with fixed point theorems does not hold.

- WE can be defined equivalently in terms of flow variables and binary participation variables:

$$(f_j^*, z_j^*) \in \arg \max_{\substack{f_j^p \geq 0,\ z_j \in \{0,1\} \\ \sum_p f_j^p = t_j,\ \text{if } z_j = 1}} \left\{ z_j t_j - \sum_p \left(l^p(f^p) + q^p \right) f^p \right\}$$

- In view of the Wardrop assumption, this converts the problem into a mixed integer-linear program and yields an equivalent characterization of a WE:
  - Positive flows on minimum effective cost paths
  - If $z_j = 0 \rightarrow f_j^p = 0$, $\forall\ p$; if $z_j = 1 \rightarrow \sum_p f_j^p = t_j$.
  - If $\min_p \{l^p(f^p) + q^p\} < 1 \rightarrow z_j = 1\ \forall\ j$.

Example

- There does not exist a WE at all price vectors p.

Example:

- At price 0, there is no WE.
- At price 0.5, there is a WE in which A sends his flow, B does not.
- This is indeed the profit maximizing price.
- Consider the social problem for this example:

$$\text{maximize}_{x_A, x_B} \{u_A(x_A) + u_B(x_B) - l(x_A + x_B)(x_A + x_B)\}$$

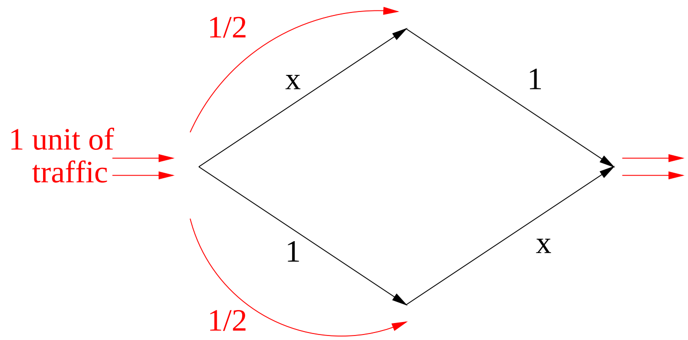
  – At the social optimum, A sends his flow, B does not.

Our results

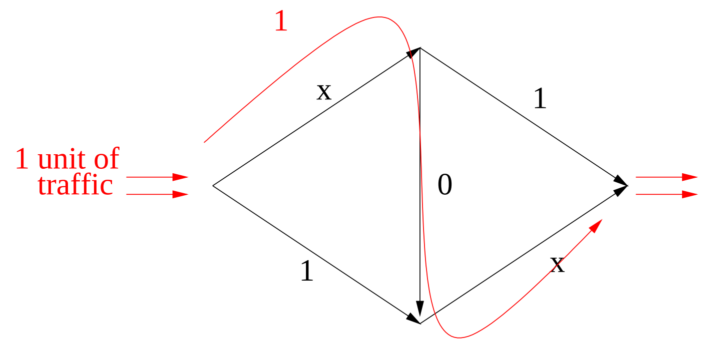
- Consider a general topology network
- Assume that l^e is continuous and strictly increasing.
  - For a given price $q \geq 0$, if there exists a WE, it is unique.
- There exists a profit maximizing price at which there is a WE:
  - There exists a monopoly equilibrium.
- The flow allocation at the ME is identical to the social optimum.
- Entire user surplus extracted (special feature of monopoly).
- One interesting question is to look at the multiple provider case, where user surplus is positive.

Braess Paradox

- Idea: Addition of an intuitively helpful link negatively impacts users of the network



C_{eq} = 1/2 (1/2+1) + 1/2 (1/2+1) = 3/2
C_{sys} = 3/2



C_{eq} = 1 + 1 = 2
C_{sys} = 3/2

- Introduced in transportation networks [Braess 68], [Dafermos, Nagurney 84]
  - Studied in the context of communication networks, distributed computing, queueing networks [Altman et al, 03]
- Motivated research in methods of upgrading networks without degrading network performance
  - Leads to limited methods under various assumptions.

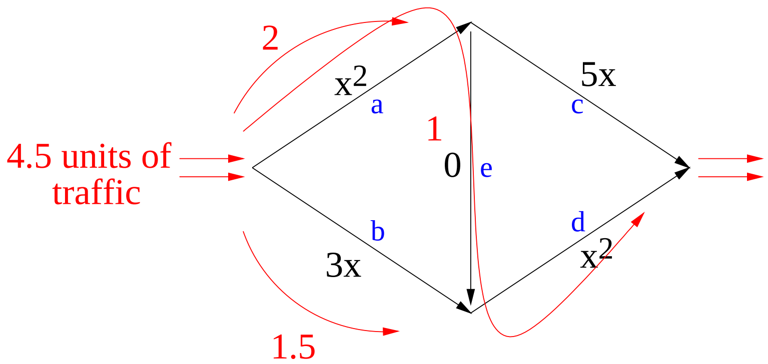
Generalized Braess Paradox

- No prices: Addition/deletion of a link one form of traffic restriction
  - [Hagstrom, Abrams 01] Let f be a WE. A Braess paradox occurs if ∃ another flow distribution ("Braess distribution") $\tilde{f}$ st
  $$l^p(\tilde{f}) \leq l^p(f), \qquad \forall\ p,$$
  $$l^{p'}(\tilde{f}) < l^{p'}(f), \qquad \text{for some } p',$$
  where $l^p(f)$: latency cost of path p under flow f.
  - If WE is a social optimum, then there is no Braess paradox.
  - Braess distribution has lower total cost than WE.
- With prices:
  - Both remarks are not true [WE does not equalize latency costs].
  - Above condition need not always constitute a paradoxical situation when you consider flows switching from one path to another.
  - Need a new definition of Braess paradox.

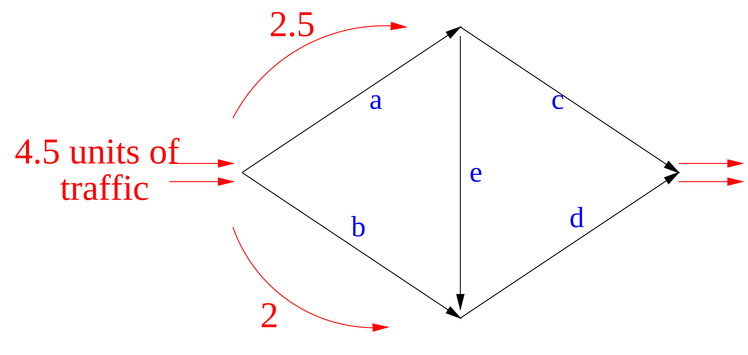
Braess Paradox with Prices

- Given a price q, let f be a WE, and $l(f) = [l^1(f), \cdots, l^{|\mathcal{P}|}(f)]$ be the path latency vector.
- Strong Braess Paradox: A BP occurs if $\exists$ some other distribution of flows, $\overline{f}$, and a transformation Δ such that $\Delta \cdot f = \overline{f}$, $\Gamma_{k,j} = \overline{\Gamma}_{k,j}$,
$$l^j(f) \geq l^i(\overline{f}), \text{ if } \Delta_{i,j} \neq 0$$
(with strict inequality for some i, j), where $\Delta_{i,j}$ is the (i,j) entry of matrix Δ.
- Remark: $\Delta_{i,j} f^j$: amount of flow moved from path j to path i.
- Weak Braess Paradox: A BP occurs if $\exists$ some other distribution of flows, $\overline{f}$, and a transformation Δ such that $\Delta \cdot f = \overline{f}$, $\Gamma_{k,j} = \overline{\Gamma}_{k,j}$,
$$l^p(f) \geq \Delta'_p \cdot l(\overline{f}), \forall\ p,$$
(with strict inequality for some $\tilde{p}$), where Δ_p is the pth column of Δ.
- Remarks:
  - $\Delta_p f^p$: vector of redistribution of flow on path p.
  - $\Delta'_p l(\bar{f})$: avg latency seen by redist. of flow on path p under $\bar{f}$.

Strong and Weak Braess Paradox



WE



Braess distribution

- WE: $l^{\{a,c\}} = 19;\ l^{\{b,d\}} = 10.75;\ l^{\{a,e,d\}} = 15.25,$
- Bd: $l^{\{a,c\}} = 18.75;\ l^{\{b,d\}} = 10;\ l^{\{a,e,d\}} = 10.25$

$$18.75 < 19;\ 10 < 10.75$$
$$0.5 \times 18.75 + 0.5 \times 10 = 14.375 < 15.2$$

- Weak Braess paradox occurs, Strong Braess paradox does not occur.

Monopoly Pricing and Braess Paradox

- Checking whether Strong/Weak Braess paradox does not occur is hard (need to consider all possible redistributions of flows)
- Proposition: Weak Braess paradox does not occur under monopoly prices.

  → Strong Braess paradox does not occur under monopoly prices.
- Intuition:
  - The sp, by setting profit maximizing prices, extracts the user surplus.
  - If there were a Braess distribution (WE not pareto optimal), the sp could extract the additional surplus and make more profit.

Extensions

- Efficient computational methods for ME.
- Multiple provider case $\longrightarrow$ interesting efficiency results for the case of competition in congested markets.
- Different routing paradigms (selfish routing vs decentralized routing for a systemwide objective).
- Pricing for differentiated services.