Index Theory for Global Uniqueness of Critical Points

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Introduction

- We are interested in establishing sufficient local conditions for global uniqueness of:
  - solutions of variational inequality problems (e.g., uniqueness of pure strategy Nash equilibrium)
  - stationary points for optimization problems
- Motivated by recent network control models which lead to nonconvex formulations mainly due to:
  - Transmission medium characteristics
  - Interaction between heterogeneous agents and protocols
- Standard convexity arguments or strict diagonal concavity conditions for uniqueness do not hold for these problems
- This talk presents index theory tools to study global uniqueness problem

Our Work

- Our objective is to establish local conditions around critical points that imply global uniqueness
  - In other work, apply these tools to study network equilibria.
- Natural Tool: Index theory of differential topology
- Poincare Hopf (PH) Theorem: relates the local properties of a vector field (around its zeros) to the topological characteristics of the underlying region, which is assumed to be a smooth manifold with boundary, under boundary conditions.
- **This talk:**
  - Generalized PH theorem for compact nonsmooth regions without boundary conditions.
    - Global uniqueness of solutions of VIs under nondegeneracy
  - Relaxing nondegeneracy assumptions for VIs under assumptions on the normal map

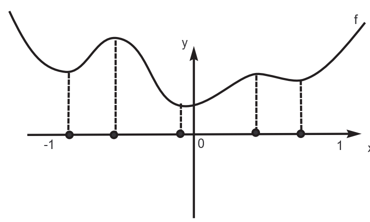
Related Work

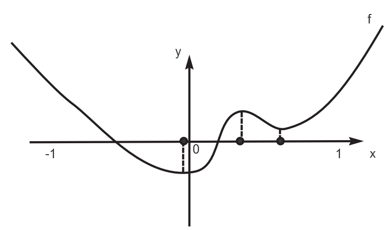
- Global Univalence of Smooth Mappings
  - Gale and Nikaido 65 (P-matrix properties)
  - Mas Colell 79 (extension to compact convex sets)
    - ► Both require global conditions on the Jacobian of the mapping
- Sensitivity and Stability Analysis of Variational Inequalities
  - Robinson 80 (Strong regularity theory for generalized equations)
  - Robinson 92 (Lipschitzian homeomorphism property for affine VI's)
  - Gowda-Pang 94 (Stability analysis via mixed NCP/degree theory)
    - ► This line of analysis establishes local uniqueness of solutions.

Intuition for the PH Theorem

- Consider a 1D smooth *non-degenerate* function (at all stationary points $x^* \in K = \{x \mid \nabla f(x) = 0\}$, $\nabla^2 f(x^*)$ is nonsingular) which is increasing at the boundary of its region.
- Observation: (# of local minima of f)= (# of local maxima of f)+1:

$$\sum_{x \in K} \text{sign}(\nabla^2 f(x)) = 1$$





Classical PH Theorem

- Let $M \subset \mathbb{R}^n$ be an n-dimensional compact smooth manifold with boundary, and $F : M \to \mathbb{R}^n$ be a continuously differentiable function. Let $Z(F, M) = \{x \in M \mid F(x) = 0\}$ denote the set of zeros of F over M. Assume the following:
  - F points outward on the boundary of M.
  - Every $x \in Z(F, M)$ is a non-degenerate *zero* of F.

  Then, the sum of Poincare-Hopf indices corresponding to zeros of F over M equals the Euler characteristic of M, $\chi(M)$. In other words,

$$\chi(M) = \sum_{x \in Z(F,M)} \mathrm{sign}(\det(\nabla F(x))).$$

- Euler characteristic is a topological invariant of sets:
  - Let M be a nonempty compact convex set, then $\chi(M) = 1$.
  - Let $S^n = \{x \mid \|x\| = 1\}$, then $\chi(S^n) = 2$ for n even and $\chi(S^n) = 0$ for n odd.

Restrictions

- Applies to $n-$dimensional compact smooth manifold with boundary.
  - A region $M = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$, where g is a continuously differentiable function is a smooth manifold with boundary.
  - A region $M = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i = 1, \ldots, I\}$ need not be a smooth manifold even when the g_j are continuously differentiable.
- Boundary assumptions too restrictive for both optimization and equilibrium problems in networks.

Towards Generalizing PH Theorem

- Consider a compact region $M = \{x \mid g_i(x) \leq 0,\ i = 1, \ldots, I\}$, where g_i are twice continuously differentiable and $I(x) = \{i \mid g_i(x) = 0\}$. Assume every $x \in M$ satisfies LICQ; the vectors $\{\nabla g_i(x) \mid i \in I(x)\}$ are linearly independent.

- Definition: Let $F : M \to \mathbb{R}^n$ be continuously differentiable.

  (a) $x \in M$ is a generalized critical point of F if $-F(x) \in N_M(x)$ [denoted by $\mathrm{Cr}(F, M)$].

  (b) Let $\lambda(x) \geq 0$ be the unique vector s.t.

  $$F(x) + \sum_{i \in I(x)} \lambda_i(x) \nabla g_i(x) = 0.$$

  The vector $x \in M$ is non-degenerate if $-F(x) \in \mathrm{ri}(N_M(x))$.

  (c) Define

  $$\Gamma(x) = V(x)^T (\nabla F(x) + \sum_{i \in I(x)} \lambda_i(x) \nabla^2 g_i(x)) V(x),$$

  where $V(x) = [v_j]_{j \notin I(x)}$ is an orthonormal basis of the tangent space at x. The vector x is non-singular if $\Gamma(x)$ is a non-singular matrix.

Our Generalization of PH Theorem

- Definition: Let $x \in \mathrm{Cr}(F, M)$ and assume that x is non-degenerate and non-singular. We define the index of F at x as

$$\mathrm{ind}_F(x) = \mathrm{sign}(\det(\Gamma(x))).$$

- Theorem: Assume that every $x \in \mathrm{Cr}(F, M)$ is non-degenerate and non-singular. Then, F has a finite number of critical points over M and

$$\sum_{x \in \mathrm{Cr}(F,M)} \mathrm{ind}_F(x) = \chi(M).$$

- No boundary conditions.
- Applies to generalized critical points.

Proof

Based on local extension of region M to a smooth manifold.

Steps of the Proof:

1. Projection of proximal points on a nonconvex set:

- Define the set $M^\epsilon = \{x \in \mathbb{R}^n | \ \|x - y\| < \epsilon \ \text{ for all } y \in M\}$ and the projection correspondence $\pi : \mathbb{R}^n \to M$ as

$$\pi(y) = \arg\min_{x \in M} \|y - x\|.$$

- We show that for ϵ small, π is a Lipschitz function (under LICQ).
- We further characterize the set over which the projection is continuously differentiable and derive an explicit expression for the Jacobian of the projection.

Proof - Continued

2. Extension Theorem: Let $F_K : \text{cl}(M^\epsilon) \to \mathbb{R}^n$ be defined as

$$F_K(y) = F(\pi(y)) + K(y - \pi(y)).$$

Then,

(i) F_K is a continuous function.

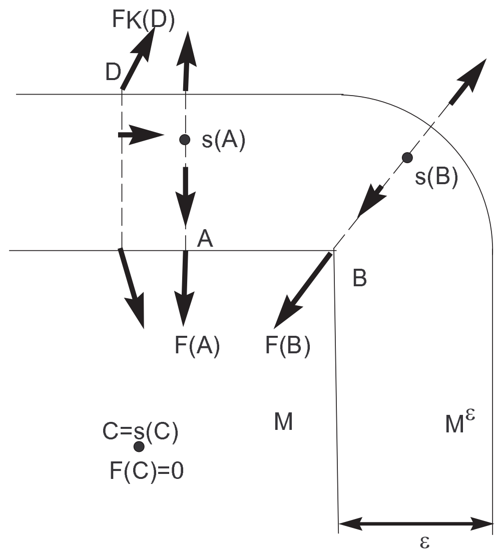
(ii) For K large, F_K points outward on the boundary of $\text{cl}(M^\epsilon)$.

(iii) There exists a 1-1 correspondence between the zeros of F_K over $\text{cl}(M^\epsilon)$, denoted Z, and the critical points of F over M, i.e., there exists a 1-1 and onto function $s : \text{Cr}(F, M) \to Z$.

(iv) If $x \in \text{Cr}(F, M)$, then F_K is continuously differentiable at $s(x)$ and

$$\text{ind}_F(x) = \text{sign}(\det(\nabla F_K(s(x)))).$$

Intuition for the Extension Theorem



A, B, C are the generalized critical points of F, whereas $s(A), s(B)$, and $s(C)$ are zeros of the extended function F_K.

3. $\text{cl}(M^\epsilon)$ is a smooth manifold with boundary and is homotopy equivalent to M, i.e., $\chi(\text{cl}(M^\epsilon)) = \chi(M)$.

- Apply classical PH Theorem to F_K and $\text{cl}(M^\epsilon)$.

Variational Inequality Problem

Definition: Let $F : M \to \mathbb{R}^n$ be a function. The *variational inequality problem* is to find a vector $x \in M$ such that

$$(y - x)^T F(x) \geq 0, \qquad \forall\, y \in M.$$

We denote the set of solutions to this problem with $\mathrm{VI}(F, M)$.

• When $M \subset \mathbb{R}^n$ is a closed convex region given by finitely many inequality constraints:

$$x \in \mathrm{VI}(F, M) \iff x \in \mathrm{Cr}(F, M).$$

Proposition: Let $F : M \to \mathbb{R}^n$ be a continuously differentiable function. Assume that every $x \in \mathrm{VI}(F, M)$ is non-degenerate and non-singular, and $\mathrm{ind}_F(x) = 1$ for all $x \in \mathrm{VI}(F, M)$. Then, $\mathrm{VI}(F, M)$ has a unique element.

Two Implications

- Uniqueness of global minimum in optimization problems only depending on local conditions.
  - Determinant of projected Hessian at KKT points positive (related uniqueness results by Jongen *et al.*).
- Uniqueness of pure strategy Nash Equilibrium, which under regularity conditions, generalize Rosen's conditions.
- Non-degeneracy assumption restrictive, hard to check.

Relaxing the Non-degeneracy Assumption

- Relation between solutions $\mathrm{VI}(F, M)$ and zeros of the normal map

Definition: The normal map associated with the variational inequality problem defined by (F, M), $F_M^{\mathrm{nor}} : \mathbb{R}^n \to \mathbb{R}^n$, is given by

$$F_M^{\mathrm{nor}}(z) = F(\pi_M(z)) + z - \pi_M(z).$$

Lemma: A vector x belongs to $\mathrm{VI}(F, M)$ iff there exists a vector z such that $x = \pi_M(z)$ and $F_M^{\mathrm{nor}}(z) = 0$.

Definition: Let $x \in \mathrm{VI}(F, M)$ and $z = x - F(x)$ be the corresponding zero of F_M^{nor}.

(a) x has the injective normal map (INM) property if F_M^{nor} is injective in a neighborhood of z.

(b) Let x be non-singular with the INM property. We define the index of F at x as

$$\mathrm{ind}_F(x) = \mathrm{sign}(\det(\Gamma(x))).$$

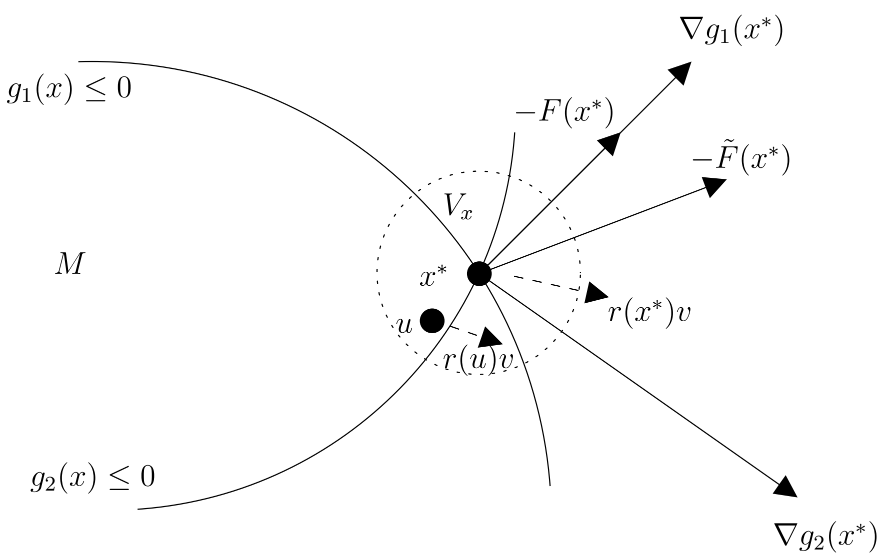
Perturbation Theorem

Theorem: Let $x^* \in \mathrm{VI}(F, M)$ be a degenerate solution which has the INM property. Then, there exists a function $\tilde{F} : M \to \mathbb{R}^n$ such that

(a) The solution set is unchanged, i.e., $\mathrm{VI}(\tilde{F}, M) = \mathrm{VI}(F, M)$.

(b) x^* is a non-degenerate solution of problem $(\tilde{F}, M)$.

(c) $\mathrm{ind}_F(x^*) = \mathrm{ind}_{\tilde{F}}(x^*)$, i.e. the index of x^* as the degenerate solution of $\mathrm{VI}(F, M)$ is equal to the index of x^* as the non-degenerate solution of $\mathrm{VI}(\tilde{F}, M)$.



Strong Stability of a Solution

Definition (*Strong Stability*): The vector $x \in \mathrm{VI}(F, M)$ is strongly stable if for every neighborhood $\mathcal{N}$ of x with $\mathrm{VI}(F, M) \cap \mathcal{N} = \{x\}$, there exist $c, \epsilon > 0$ s.t., for any two functions $G, H \in \mathbb{B}(F, \epsilon, M \cap \mathrm{cl}\mathcal{N})$,

$$\mathrm{VI}(G, M) \cap \mathcal{N} \neq \emptyset, \;\; \mathrm{VI}(H, M) \cap \mathcal{N} \neq \emptyset$$

and for every $x' \in \mathrm{VI}(G, M) \cap \mathcal{N}$ and $x'' \in \mathrm{VI}(H, M) \cap \mathcal{N}$,

$$\|x' - x''\| \leq c\|e_G(x') - e_H(x'')\|,$$

where $e_G(x) = F(x) - G(x)$ and $e_H(x) = F(x) - H(x)$.

Notation: Let the index sets $\alpha(x)$ and $\beta(x)$ be given by

$$\alpha(x) = \{i \in I \mid \lambda_i(x) > 0 = g_i(x)\},$$
$$\beta(x) = \{i \in I \mid \lambda_i(x) = 0 = g_i(x)\},$$

and let $\mathcal{B}(x)$ be the set of matrices defined by

$$\mathcal{B}(x) = \Big\{ B \mid B = [\nabla g_i(x)]_{i \in J}, \; \alpha(x) \subseteq J \subseteq \alpha(x) \cup \beta(x) \Big\}.$$

INM Property and Strong Stability

Theorem: Let $x^* \in \mathrm{VI}(F, M)$.

(a) If x^* is strongly stable, then it satisfies the INM property.

(b) x^* is strongly stable iff all matrices of the form

$$V_B^T\Big(\nabla F(x) + \sum_{i \in I(x)} \lambda_i(x) \nabla^2 g_i(x)\Big) V_B$$

have the same nonzero determinantal sign, where $B \in \mathcal{B}(x)$, and V_B is a matrix whose columns form an orthonormal basis for the nullspace of B.

(c) If x^* is non-degenerate and non-singular, then it is strongly stable.

Example: Let $M = [-1, 0] \subset \mathbb{R}^n$ and $F(x) = -x^2$. Then, $VI(F, M)$ has a unique degenerate solution at $x^* = 0$. Moreover, we have

$$F_M^{\mathrm{nor}}(z) = \begin{cases} -z^2, & \text{if } z \in [-1, 0], \\ z, & \text{otherwise.} \end{cases}$$

Hence x^* has the INM property. However, x^* is not a strongly stable solution.

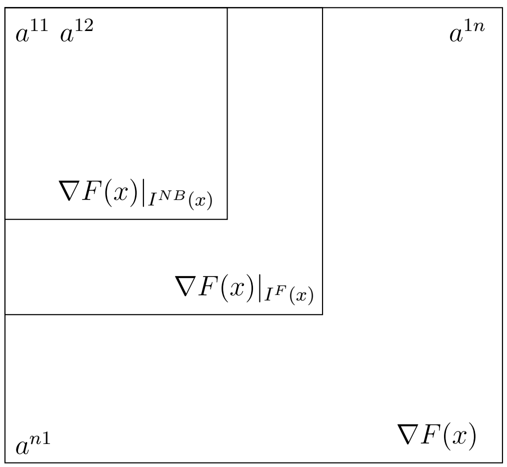
Matrix Condition for Box-Constrained Regions

- Consider the special case of $M = [a, b]$. Define

$$I^{NB}(x^*) = \Big\{ i \in \{1, .., n\} \mid a_i < x_i^* < b_i \Big\},$$
$$I^F(x^*) = \Big\{ i \in \{1, .., n\} \mid F_i(x^*) = 0 \Big\}$$

- The matrix condition equivalent to checking positivity of certain principal minors of the Jacobian of F:

  $\det(\nabla F(x^*)|_J) > 0$ for all J such that $I^{NB}(x^*) \subset J \subset I^F(x^*)$.



- “Partial P-matrix” property of the Jacobian

Global Index Theorem for Variational Inequalities

Theorem: Assume that every $x \in \mathrm{VI}(F, M)$ is non-singular and has the INM property. Then $\mathrm{VI}(F, M)$ has a finite number of elements. Moreover:

$$\sum_{x \in \mathrm{VI}(F,M)} \mathrm{ind}_F(x) = 1.$$

- Assume that for every $x \in \mathrm{VI}(F, M)$,

$$\det\left(V_B^T\Big(\nabla F(x) + \sum_{i \in I(x)} \lambda_i(x) \nabla^2 g_i(x)\Big) V_B\right) > 0,$$

  for all $B \in \mathcal{B}(x)$ and V_B. Then $\mathrm{VI}(F, M)$ has a unique element.

Applications

Our results used in recently studied network control models and oligopoly models of competition:

- Wireless power control in the presence of interference, studied by Huang-Berry-Honig, 2005.
- Heterogeneous congestion control protocols in a wireline network, studied by Tang-Wang-Low-Chiang, 2005.
- Cournot equilibrium, studied by Kolstad-Mathiesen, 1987.
- Equilibrium of price competition in the presence of congestion externalities, studied by Acemoglu-Ozdaglar, 2005.

Conclusions

- A generalized PH Theorem for compact nonsmooth regions represented by finitely many inequality constraints.
- Local indices for degenerate solutions of VIs under assumptions on the normal map.
  - Global uniqueness of solutions to general VIs
- Nonconvexities becoming more prominent in network control models.
- Theory of VIs plays a key role in their analysis.

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