Applications of Generalized Poincare-Hopf in Network Optimization

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Introduction

- Two Related Problems: Establishing sufficient (local) conditions for:
  - uniqueness of stationary points for optimization problems
  - uniqueness of solutions of variational inequality problems (e.g., uniqueness of pure strategy Nash equilibrium)
- Standard approach is to use (strict) convexity for the optimization and strict diagonal concavity for the equilibrium problem [Rosen].
- Too restrictive since recent network control models lead to nonconvex formulations mainly due to two reasons:
  - Nonlinear dependencies between control variables,
  - Interaction between heterogeneous agents and heterogenous same/cross layer protocols result in typically nonconvex equilibrium problems.
- This talk presents differential topology tools to analyze network optimization/equilibrium problems.

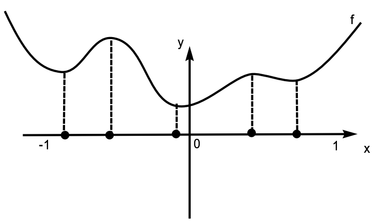
Our Approach

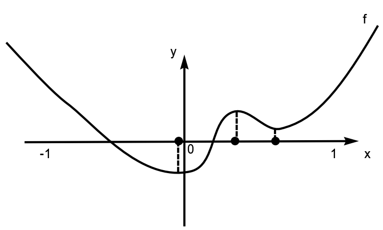
- Our objective is to establish local conditions around equilibrium points that imply uniqueness of an equilibrium and show its use in network models.
- Natural Tool: Index theory of differential topology
- Poincare Hopf (PH) Theorem: relates the local properties of a vector field (around its zeros) to the topological characteristics of the underlying region, which is assumed to be a smooth manifold with boundary under boundary conditions.
- We prove a generalized PH theorem for compact nonsmooth regions for generalized equilibria without boundary conditions [Alp's talk today].
- This talk, focus on the special case of “box-constrained regions”.
- Use the PH Theorem:
  - for the gradient of the objective function of an opt. problem.
  - for the function defining the variational inequality problem.

Intuition for the PH Theorem

- Consider a 1D smooth *non-degenerate* function (at all stationary points $x^* \in K = \{x \mid \nabla f(x) = 0\}$, $\nabla^2 f(x^*)$ is nonsingular) which is increasing at the boundary of its region.
- Observation: (# of local minima of f)= (# of local maxima of f)+1:

$$\sum_{x \in K} \text{sign}(\nabla^2 f(x)) = 1$$





Our Generalization for Box Constrained Problems

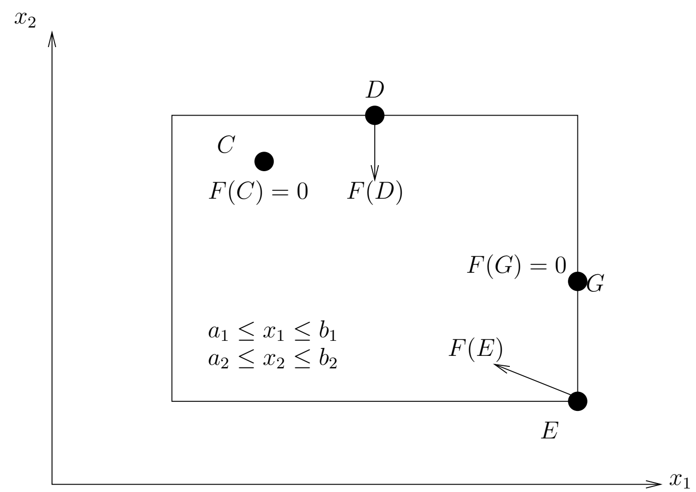
- Let $M = [a, b] \subset \mathbb{R}^n$ and $F : M \mapsto \mathbb{R}^n$.
- *Mixed Complementarity Problem (MCP)* is to find $x \in M$ s.t. for each $i \in \{1, .., n\}$:

$$
\begin{aligned}
x_i &= a_i, \qquad F_i(x) \geq 0 && (1)\\
a_i < x_i &< b_i, \qquad F_i(x) = 0. && (2)\\
x_i &= b_i, \qquad F_i(x) \leq 0 && (3)
\end{aligned}
$$

We denote the set of solutions to (1)-(3) by MCP$(F, [a, b])$. This set is equivalent to

- the set of KKT points (for minimization) of f over M when $F = \nabla f$.
- the solution set for the variational inequality problem defined by F over M.
- the set of "generalized equilibria" of F over M.

Our Generalization for Box Constrained Problems



Def: We say that $x^* \in \mathrm{MCP}(F, M)$ is a *complementary solution* if $x_i^* = a_i$ implies $F_i(x^*) > 0$ and $x_i^* = b_i$ implies $F_i(x^*) < 0$, i.e. $I^{NB-\mathrm{MCP}}(x^*) = I^F(x^*)$, where $I^F(x^*) = \Big\{i \in \{1,..,n\} \mid F_i(x^*) = 0\Big\}$ and $I^{NB-\mathrm{MCP}}(x^*) = \Big\{i \in \{1,..,n\} \mid a_i < x_i^* < b_i\Big\}$.

Assumption (SCS): Each $x \in \mathrm{MCP}(F, M)$ is a complementary solution.

Notation: For an $n \times n$ matrix and $J \subset \{1, \ldots, n\}$, let $A|_J$ denote the principal sub-matrix that contains $A^{i,j}$ with $i, j \in J$.

Our Generalization for Box Constrained Problems

Theorem: Let $F : M \mapsto \mathbb{R}^n$ be a continuously differentiable function. Assume that (F, M) satisfies Assumption SCS. Moreover, assume that for each vector $x^* \in \mathrm{MCP}(F, M)$,

$$\det(\nabla F(x^*)|_{I^{NB-\mathrm{MCP}}(x^*)}) \neq 0.$$

Then, $\mathrm{MCP}(F, M)$ has a finite number of elements and

$$\sum_{x^* \in \mathrm{MCP}(F,M)} \mathrm{sign}(\det(\nabla F(x^*)|_{I^{NB-\mathrm{MCP}}(x^*)}) = 1.$$

Proof: Based on local extension of region M to a smooth manifold and appropriately extending F on this region such that classical PH can be applied.

Relaxing the Regularity Conditions

Corollary Let $F : M \mapsto \mathbb{R}^n$ be a continuously differentiable function. Assume that (F, M) satisfies Assumptions SCS and $\forall x^* \in \text{MCP}(F, M)$,

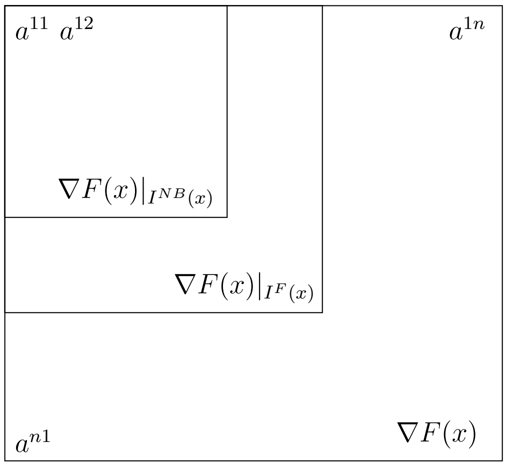
$$\det(\nabla F(x^*)|_{I^{NB-MCP}(x^*)}) > 0.$$

Then, $\text{MCP}(F, M)$ has a unique element.

- Assumption (SCS) difficult to establish.
- We relax it for box constrained regions.

Relaxing SCS

Assumption (Strong Non-degeneracy (SND)): Each $x^* \in \text{MCP}(F)$ is *strongly non-degenerate*, i.e., $\det(\nabla F(x^*)|_J) > 0$ for all J such that $I^{NB-MCP}(x^*) \subset J \subset I^F(x^*)$.



Theorem: Assume that (F, M) satisfies Assumption SND. Then, $\text{MCP}(F)$ has a unique element.

Proof is based on exploiting properties of *partial P-matrices*.

Partial P-matrices

Definition: An $n \times n$ matrix A is called a *P-matrix* if the determinant of each of its principal sub-matrices is positive, i.e. if

$$\det(A|_J) > 0, \qquad \forall\ J \subset \{1, 2, .., n\}.$$

- P-matrices play an important role in establishing *global univalance* of continuous maps (see the celebrated Gale-Nikaido Theorem).
- Weaker than positive definiteness when matrix isn't symmetric.

Definition: Given an index set $I \subset \{1, 2, .., n\}$ and an $n \times n$ matrix A, we say that A is a *partial P-matrix with respect to I* if

$$\det(A|_J) > 0, \qquad \forall\ J \text{ with } I \subset J \subset \{1, 2, .., n\}.$$

- Every P-matrix is a partial P-matrix wrt any $I \subset \{1, .., n\}$.
- F satisfies Assumption SND if and only if $\nabla F(x^*)|_{I^{F}(x^*)}$ is a partial P-matrix with respect to $I^{NB-MCP}(x^*)$ for all $x^* \in \text{MCP}(F, M)$.

Applications

Corollary: Let $f : U \mapsto \mathbb{R}$ be twice continuously differentiable where $M = [a, b] \subset U$. Assume that $(\nabla f, M)$ satisfies Assumption SND, i.e., for each $x \in \text{KKT}(f, M)$, we have $\nabla^2 f(x)|_{I^{\nabla f}(x)}$ is a partial P-matrix wrt $I^{NB}(x)$. Then, $\text{KKT}(f, M)$ has a unique element which is the unique local (global) minimum.

Use our results in recently studied network control models and oligopoly models of competition:

- Wireless power control in the presence of interference, studied by Huang-Berry-Honig, 2005.
- Heterogeneous congestion control protocols in a wireline network, studied by Tang-Wang-Low-Chiang, 2005.
- Cournot equilibrium, studied by Kolstad-Mathiesen, 1987.
- Equilibrium of price competition in the presence of congestion externalities, studied by Acemoglu-Ozdaglar, 2005.

Wireless Power Control Problem

- Single-hop power control in the presence of interference.

  Model: Let $L = \{1, 2, ..., n\}$ denote the set of nodes and

$$\mathcal{P} = \prod_{i \in L} [P_i^{\min}, P_i^{\max}] \subset \mathbb{R}^n$$

  denote the set of power vectors p such that each node $i \in L$ transmits at a power level p_i.

- Received SINR for each node i, $\gamma_i : \mathcal{P} \mapsto \mathbb{R}$

$$\gamma_i(p) = \frac{p_i h_{ii}}{n_0 + \sum_{j \neq i, 1 \leq j \leq n} p_j h_{ji}}$$

- The power control problem is:

$$\min_{p \in \mathcal{P}} f(p) = \quad - \sum_{1 \leq i \leq n} u_i(\gamma_i(p)) \quad .$$

Uniqueness for the Power Control Problem

- The objective function of the power control prob is nonconvex in p.

Proposition: Let

$$\gamma_i^{\min} = \min_{p \in \mathcal{P}} \gamma_i(p), \qquad \gamma_i^{\max} = \max_{p \in \mathcal{P}} \gamma_i(p),$$

and assume that each utility function satisfies the following assumption regarding its *coefficient of relative risk aversion*:
(A) $-\frac{\gamma_i u_i''(\gamma_i)}{u_i'(\gamma_i)} \in [1, 2], \qquad \forall\ \gamma_i \in [\gamma_i^{\min}, \gamma_i^{\max}].$
Then, the power control problem has a unique KKT point.

Proof: Under the given assumptions, show that $\nabla^2 f(p)|_{I^{\nabla f}(p)}$ is a P-matrix for all $p \in KKT(f, \mathcal{P})$.

Heterogenous Congestion Control Protocols

Utility-based framework used by Kelly et. al.(1998), Low (1999) to study homogeneous protocols. We consider the heterogenous model studied by Tang et al (2005):

- $\mathcal{L}$ is the set of links with finite capacities c_l, $\mathcal{J}$ is the set of protocols, $u_s^j(z)$ denotes utility of user (s, j) from sending z units of traffic.
- The effective price for protocol j on link l is denoted by a function $m_l^j(p_l)$. The price observed by a user is $q_s^j(p)$.
- The traffic sent by each user is given by

$$x_s^j(p) = \text{argmax}_{z \geq 0} u_s^j(z) - z q_s^j(p).$$

- $y^l(p)$ denotes the total flow on link l and the equilibrium set is given by

$$E = \{p |\ p \geq 0,\ y^l(p) \leq c_l,\ p_l(y^l(p) - c_l) = 0\ \forall\ l \in L\}$$

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Heterogenous Congestion Control Protocols

Under mild assumptions, $E \subset \mathcal{P} = [0, p^{\max}]$. Let $F_l(p) = c_l - y_l(p)$.

- Tang et al uniqueness argument: Assumptions that guarantee $p_l > 0$ for all $p \in E$ (i.e., every link is congested in equilibrium, strong!):
  - E is the same as the set of zeros of F over $\mathcal{P}$, and using classical PH, there is a unique equilibrium if $\det(\nabla F(p)) > 0$ for each $p \in Z = \{p \mid F(p) = 0\}$.
- Our generalization: Allow for excess capacity (zero price) in equilibrium (generalized equilibrium):
  - E coincides with $\mathrm{MCP}(F, \mathcal{P})$. There is a unique equilibrium if $\nabla F(p)|_{I^F(p)}$ is a partial P-matrix wrt $I^{NB}(p)$ for each $p \in \mathrm{MCP}(F, \mathcal{P})$.
- Tang et al show conditions on the underlying m and u which is sufficient for uniqueness (under the assumption that all links are congested). We show that the same conditions establish uniqueness in the more general case.

Conclusions and Future Work

- A generalized PH Theorem for box constrained regions.
- Applications on network optimization, network equilibrium models.
- Relax the regularity assumptions for more general polyhedral constraints.
- Stability properties of P-matrices.