A Unifying Framework for Duality and Minimax

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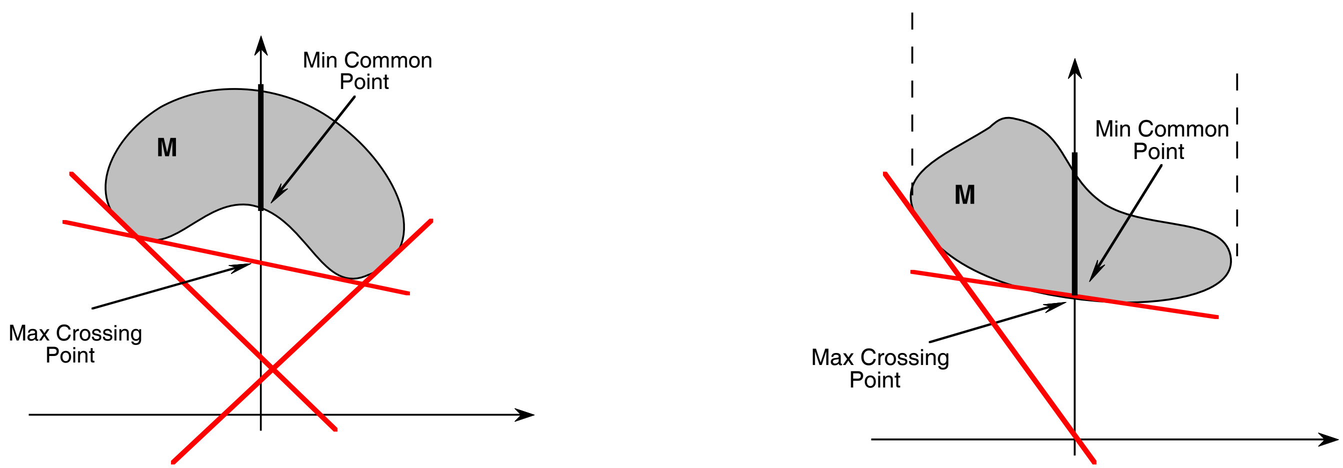
Motivation

- Minimax Theory: Given $\phi : X \times Z \mapsto \Re$, where $X \subset \Re^n$, $Z \subset \Re^m$, under what conditions do we have
$$\sup_{z\in Z} \inf_{x\in X} \phi(x,z) = \inf_{x\in X} \sup_{z\in Z} \phi(x,z) \ ?$$
- Optimization Duality: Consider the problem
$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X,\ g_j(x) \le 0,\ j = 1, \ldots, r. \end{array}$$
Define the Lagrangian function : $L(x,\mu) = f(x) + \sum_{j=1}^{r} \mu_j g_j(x)$.
  - Primal problem: $f^* = \inf_{x\in X} \sup_{\mu\ge 0} L(x,\mu)$
  - Dual problem: $q^* = \sup_{\mu\ge 0} q(\mu) = \sup_{\mu\ge 0} \inf_{x\in X} L(x,\mu)$
- All of (convex/concave) minimax theory and duality theory can be developed in terms of simple geometry of convex sets! (Need machinery from convex analysis)

Min Common/Max Crossing Problems

Let M be a nonempty subset of $\Re^{n+1}$

- Min Common Point Problem: Consider all vectors that are common to M and the $(n+1)$st axis. Find one whose $(n+1)$st component is minimum.
- Max Crossing Point Problem: Consider nonvertical hyperplanes that contain M in their "upper" closed halfspace. Find one whose crossing point of the $(n+1)$st axis is maximum.



Weak Duality

- Optimal value of the min common problem:

$$w^* = \inf_{(0,w)\in M} w.$$

- Focus on hyperplanes with normals $(\mu, 1)$ whose crossing point ξ satisfies

$$\xi \leq w + \mu' u, \qquad \forall\ (u, w) \in M.$$

- Maximum crossing level over all hyperplanes with normal $(\mu, 1)$ is

$$q(\mu) = \inf_{(u,w)\in M}\{w + \mu' u\}$$

Max crossing problem: maximize $q(\mu)$

subject to $\mu \in \Re^n$.

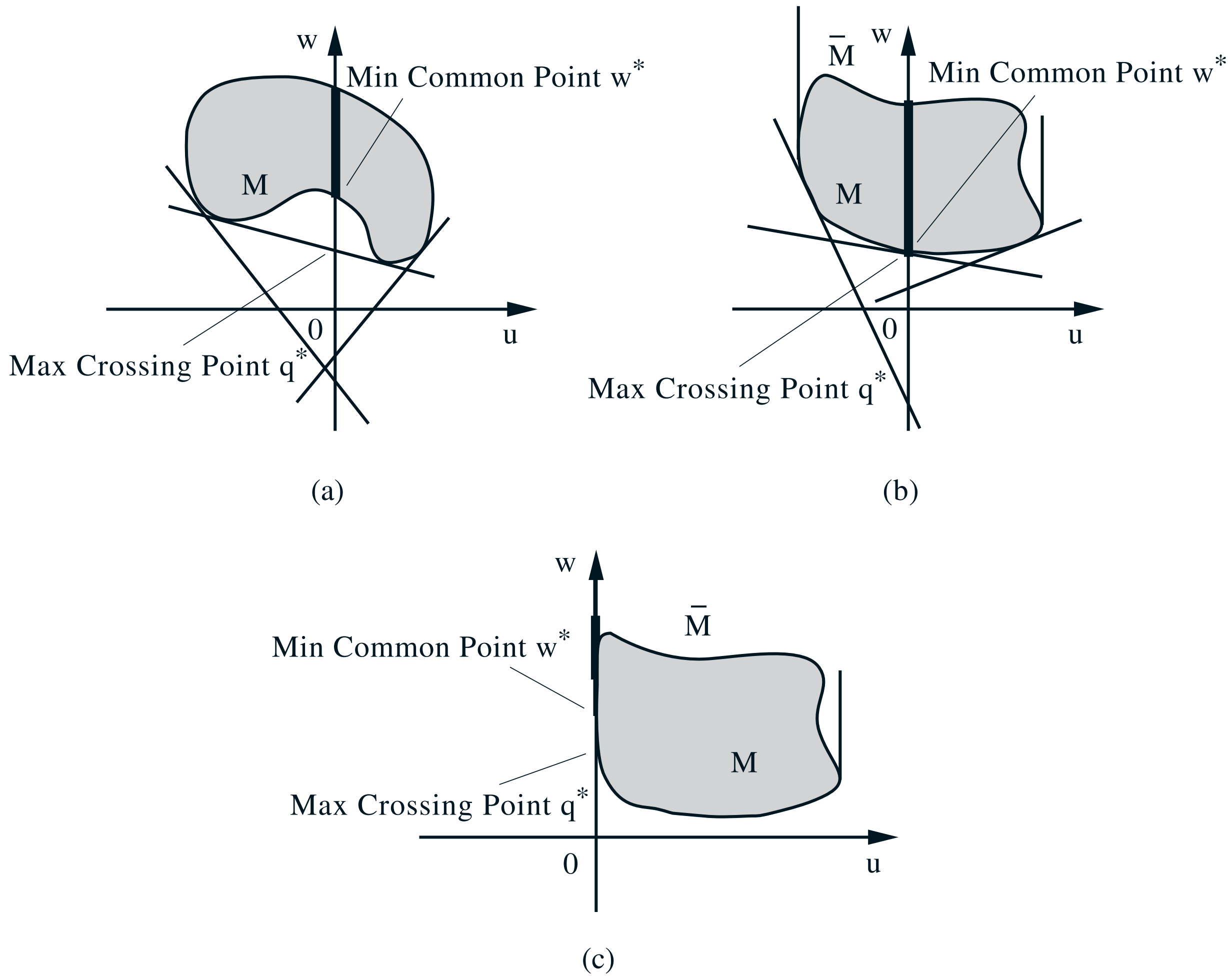
- Note that for all $(u, w) \in M$ and $\mu \in \Re^n$,

$$q(\mu) = \inf_{(u,w)\in M}\{w + \mu' u\} \leq \inf_{(0,w)\in M} w = w^*,$$

$$\sup_{\mu\geq 0} q(\mu) = q^* \leq w^*.$$

Strong Duality

Question: Under what conditions do we have $q^* = w^*$ and the supremum in the max crossing problem is attained?



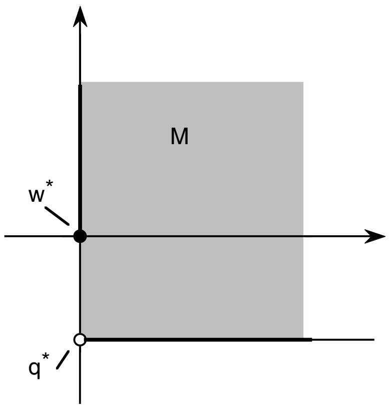
Duality Theorems

- Assume that w^* is finite and that the set

$$\bar{M} = \{(u, w) \mid \exists\ \bar{w} \text{ with } \bar{w} \le w \text{ and } (u, \bar{w}) \in M\}$$

is convex.

- Min Common Max Crossing Theorem I: We have $q^* = w^*$ if and only if for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, we have $w^* \le \liminf_{k\to\infty} w_k$.



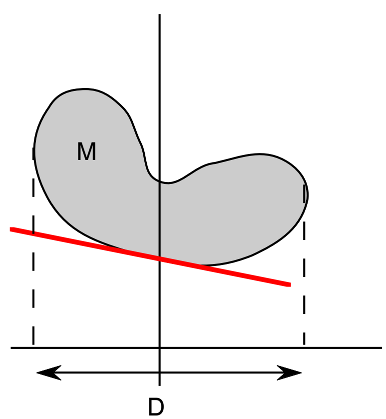
Attainment of Optimum in Max Crossing Problem

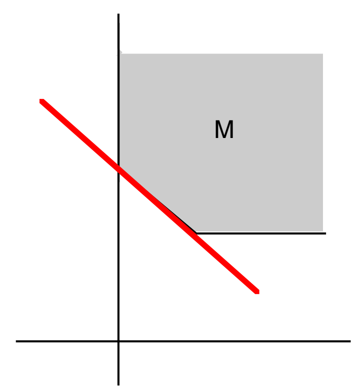
- Min Common Max Crossing Theorem II: Assume that the set

$$D = \{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in \bar{M}\}$$

contains the origin in its relative interior. Then $q^* = w^*$ and there exists a vector $\mu \in \Re^n$ such that $q(\mu) = q^*$.

- Min Common Max Crossing Theorem III: Involves polyhedral assumptions and guarantees $q^* = w^*$ as well as attainment of the optimum of the max crossing problem.





Minimax Problems

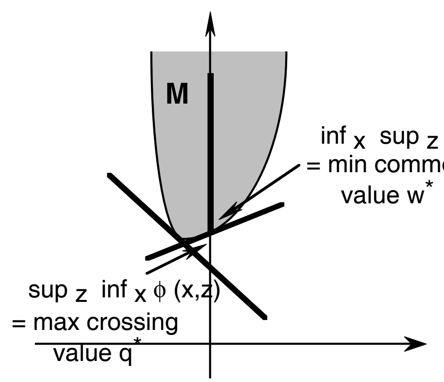
- Given $\phi : X \times Z \mapsto \Re$, where $X \subset \Re^n$, $Z \subset \Re^m$, under what conditions do we have

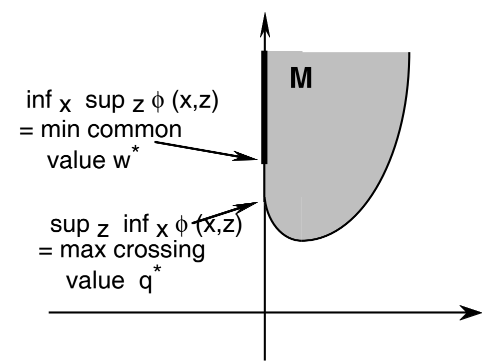
$$\sup_{z \in Z} \inf_{x \in X} \phi(x,z) = \inf_{x \in X} \sup_{z \in Z} \phi(x,z) \quad ?$$

- Introduce the perturbation function $p : \Re^m \mapsto [-\infty, \infty]$

$$p(u) = \inf_{x \in X} \sup_{z \in Z} \{\phi(x,z) - u'z\}, \qquad u \in \Re^m.$$

- Apply min common/max crossing framework with $M = \text{epi}(p)$.





- Note that $w^* = \inf_{x \in X} \sup_{z \in Z} \phi(x,z)$.
  - Convexity in x implies M is a convex set.
  - Concavity/semicont. in z implies $q^* = \sup_{z \in Z} \inf_{x \in X} \phi(x,z)$.

Minimax Theorems

- Assume that:
  - X and Z are convex and $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$ is finite.
  - For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.
  - For each $x \in X$, the function $-\phi(x, \cdot)$ is closed and convex.

- Minimax Thm. I: The minimax equality holds iff the function p is lower semicontinuous at $u = 0$.

- Minimax Thm. II: If 0 lies in the relative interior of $\mathrm{dom}(p)$, then the minimax equality holds and the supremum in $\sup_{z \in Z} \inf_{x \in X} \phi(x, z)$ is attained by some $z \in Z$.
  - Proofs by applying min common max crossing theorems to the set

$$M = \mathrm{epi}(p).$$

Conditions for Attaining the Minimum

- Assume that:

  1. X and Z are convex and $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty$.
  2. For each $x \in X$, the function $r_x(z) = -\phi(x, z)$ if $z \in Z$, and ∞ otherwise, is closed and convex.
  3. For each $z \in Z$, the function $t_z(x) = \phi(x, z)$ if $x \in X$, and ∞ otherwise, is closed and convex.
  4. The set of common directions of recession of all the functions t_z, $z \in Z$, consists of the zero vector only.

- Then, the minimax equality holds, and the infimum over X is attained at a compact set of points.

- Special cases of Condition 4:

  - X is compact.
  - $\exists$ a scalar γ and $\bar{z} \in Z$ s.t. the level set $\{x \in X \mid \phi(x, \bar{z}) \leq \gamma\}$ is nonempty and compact.

Saddle Point Theorem

- Assume that:

  1. X and Z are convex and either $-\infty < \sup_{z\in Z} \inf_{x\in X} \phi(x,z)$, or $\inf_{x\in X} \sup_{z\in Z} \phi(x,z) < \infty$.
  2. For each $x \in X$, the function $r_x(z)$ is closed and convex, and the set of common directions of recession of all the functions r_x, $x \in X$, consists of the zero vector only.
  3. For each $z \in Z$, the function $t_z(x)$ is closed and convex, and the set of common directions of recession of all the functions t_z, $z \in Z$, consists of the zero vector only.

- Then, the minimax equality holds, and the set of saddle points of ϕ is nonempty and compact.

- Special cases:

  – X and Z are compact.

  – Z is compact, and $\exists$ a scalar γ and $\bar{z} \in Z$ such that the level set $\{x \in X \mid \phi(x,\bar{z}) \leq \gamma\}$ is nonempty and compact.

Optimization Duality

- Consider the primal problem (optimal value f^*)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C, \; g_j(x) \leq 0, \; j = 1, \ldots, r, \end{array}$$

where C is a convex set, f and g_j are convex over C.

- Consider also the dual problem (optimal value q^*)

$$\begin{array}{ll} \text{maximize} & q(\mu) = \inf_{x \in C} \left\{ f(x) + \sum_{j=1}^r \mu_j g_j(x) \right\} \\ \text{subject to} & \mu \geq 0. \end{array}$$

- Main Question: Under what conditions do we have

$$f^* = q^* = q(\mu^*)?$$

- Question can be addressed using min common max crossing framework.

Nonlinear Farkas' Lemma

- Let C be convex, and f and the g_j be convex functions. Assume that

$$f(x) \geq 0, \qquad \forall\, x \in F = \{x \in C \mid g(x) \leq 0\},$$

  and one of the following conditions holds:

  1. 0 is in the relative interior of the set
$$D = \{u \mid g(x) \leq u \text{ for some } x \in C\}.$$
  2. The functions g_j, $j = 1, \ldots, r$ are affine, and $F \cap \mathrm{ri}(C) \neq \emptyset$.

- Then, there exist scalars $\mu_j^* \geq 0$, $j = 1, \ldots, r$, such that

$$f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x^*) \geq 0, \qquad \forall\, x \in C.$$

  – Reduces to Farkas' Lemma if $C = \Re^n$, and f, g_j linear.

  – Proofs by applying min common max crossing theorems to

$$M = \{(u, w) \mid \text{there is } x \in C \text{ s.t. } g(x) \leq u,\ f(x) \leq w\}.$$

Application to Convex Programming

- Consider the problem (optimal value f^*, assumed finite)

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in C,\ g_j(x) \le 0,\ j = 1, \ldots, r, \end{aligned}$$

  where C is convex, f and g_j are convex over C.

- Apply Farkas' Lemma. There exist $\mu_j^* \ge 0$ such that

$$f^* \le f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall\ x \in C.$$

  Since $F \subset C$ and $\mu_j^* g_j(x) \le 0$ for all $x \in F$,

$$f^* \le \inf_{x \in F} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} \le \inf_{x \in F} f(x) = f^*.$$

  Thus equality holds throughout above, and we have

$$f^* = \inf_{x \in C} \{ f(x) + {\mu^*}' g(x) \} = q(\mu^*).$$

Reference

Convex Analysis and Optimization,
Dimitri Bertsekas, with Angelia Nedic and Asuman Ozdaglar.

- To be published January 2003.
- http:// web.mit.edu/6.291/www-old
  - Min Common/Max Crossing Duality
  - Existence of Solutions and Strong Duality
  - Pseudonormality and Lagrange Multipliers
  - Incremental Subgradient Methods