EFFICIENCY AND BRAESS PARADOX UNDER PRICING

Asuman Ozdaglar

Joint work with
Xin Huang, [EECS, MIT], Daron Acemoglu [Economics, MIT]

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Electrical Engineering and Computer Science Dept.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Resource and Traffic Management in Communication Networks

- Flow control and routing essential components of traffic management.

- **Traditional Network Optimization:** Focus on a central objective, devise synchronous/asynchronous, centralized/distributed algorithms.
  - Assumes all users are homogeneous with no self interest

- **Today’s Large-scale Networks (eg. Internet):**
  - Decentralized operation
  - Highly heterogeneous nature of users
  - Interconnection of privately owned networks
Emerging Paradigm for Distributed Control

- Analysis of resource allocation in the presence of decentralized information, selfish users/administrative domains, and profit-maximizing service providers.

- Instead of a central control objective, model as a multi-agent decision problem.
  - Some control functions delegated to agents with independent objectives.
  - Suggests using game theory and economic market mechanisms.
Recent Literature

• Flow (congestion) control by maximizing aggregate source utility over transmission rates
  – “Kelly mechanism”: Decentralized incentive compatible resource allocation [Kelly 97], [Kelly, Maulloo, Tan 98]
  – Primal/Dual methods, stability, relations to current congestion control mechanisms [Low, Lapsley 02], [Liu, Basar, Srikant 03]

• Selfish (user-directed) routing
  – Transportation net. [Wardrop 52], [Beckmann 56], [Patriksson 94]
  – Communication networks [Orda, Rom, Shimkin 93], [Korilis, Lazar, Orda 97], [Roughgarden, Tardos 00]

• Efficiency
  – “Price of Anarchy”: Ratio of performance of selfish to performance of social [Koutsoupias, Papadimitriou 99], [Roughgarden, Tardos 00], [Correa, Schulz, Stier Moses 03], [Johari, Tsitsiklis 03]
Previous Work

- Existing literature focuses on:
  - resource allocation among competing heterogeneous users
  - social welfare (aggregate utility) maximization

- Pricing used as a means of regulating selfish user behavior and achieving social optimum in a distributed manner.

- Commercial networks operated by for-profit service providers.
  - Pricing used to make profits or provide service differentiation among users.
  - Combined study of pricing and resource allocation essential in the design of networks.

- With a few exceptions ([He, Walrand 03], [Mitra et al. 01], [Basar, Srikant 02]), this game theoretic interaction neglected.

- In [Acemoglu, Ozdaglar 04], we studied pricing with combined flow control/routing for parallel link networks.
This Talk

- Consider selfish flow choice and routing in a **general topology network** where resources are owned by for-profit entities (focus on a single service provider).
- Each user pays a price proportional to the amount of bandwidth she uses (**usage-based, linear pricing**).
- **Goal**: Develop a framework and study the implications of pricing on various performance results.
- **Two parts**:
  - Equilibrium and efficiency of combined flow control/routing
  - Braess’ paradox under pricing
Model for Decentralized System

- Directed graph $G = (V, E)$, $m$ origin destination pairs
- For each link $e$, a latency function $l^e : [0, C^e] \mapsto [0, \infty)$, where $C^e$ denotes the capacity of link $e$
  - specifies the delay on the link given its congestion.
- For each source destination pair $k$, $J_k$ set of users, $P_k$ set of paths
- For each user $j \in J_k$, a utility function $u_{k,j} : [0, \infty) \mapsto [0, \infty)$
  - measure of benefits from data transmission.
- Depending on application service requirements, utility takes different forms [Shenker 95]:
  - Inelastic applic: real time voice, video (step utility function)
  - Elastic applic: e-mail (increasing concave utility function)
- Single service provider: charges $q^p$ per unit bw on path $p$. 
User Equilibrium - Wardrop Equilibrium

• Let \( f_{k,j}^p \): flow of user \( j \in J_k \) on path \( p \).

\[
\Gamma_{k,j} = \sum_{p \in P_k} f_{k,j}^p, \quad \text{“flow rate of user \( j \)”}
\]

\[
f^p = \sum_{j \in J_k} f_{k,j}^p, \quad \text{“flow on path \( p \)”}
\]

\[
f^E = [f^1, \ldots, f^{|E|}], \quad \text{“vector of link loads”}
\]

• Let payoff function \( v_j \) of user \( j \) be defined by

\[
v_j(f_{k,j}; f^E, q) = u_{k,j}(\Gamma_{k,j}) - \sum_{p \in P_k} \sum_{e \in p} l^e(f^e) \left( f_{k,j}^p - \sum_{p \in P_k} q^p f_{k,j}^p \right).
\]

• Definition: For a given price \( q \geq 0 \), \( f^* \) is a Wardrop Equilibrium if

\[
f_{k,j}^* \in \arg \max_{f_{k,j} \geq 0} v_j(f_{k,j}; f^E, q), \quad \forall j \in J_k, \forall k,
\]

\[
f^e = \sum_k \sum_{j \in J_k} \sum_{p \mid e \in p, p \in P_k} (f^*)_{k,j}^p, \quad \forall e \in E.
\]

• Implicit Assumption: Users small relative to the network
Single Service Provider

- The SP sets prices per unit bandwidth for each path to maximize profits.
- **Monopoly Problem:**

  \[
  \text{maximize} \quad \sum_p q^p f^p(q) \\
  \text{subject to} \quad q \geq 0,
  \]

  where \( f^p(q) \) is the flow on path \( p \) at the WE given price vector \( q \).

- This problem has an optimal solution \( q^* \).

- We will refer to \( q^* \) as the **monopoly equilibrium price** and \( (q^*, f(q^*)) \) [or \( (q^*, f^*) \)] as the **monopoly equilibrium (ME)**.
Elastic Traffic

- The utility function $u_{k,j}$ is concave and nondecreasing.
- **ME price:** Let $(q, f)$ be an ME and let $\bar{J}_k = \{j \mid j \in J_k, \Gamma_{k,j} > 0\}$. Then,
  \[
  q^p = \sum_{e \in p} (l^e)'(f^e) f^e + \frac{\sum_{p \in \bar{P}_k} f^p}{\sum_{j \in \bar{J}_k} \frac{1}{u''_{k,j}(\Gamma_{k,j})}}.
  \]
- **Social Problem**
  \[
  \text{maximize } \sum_{j \in J_k} u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \bar{P}_k} \sum_{e \in p} l^e(f^e) f^p
  \]
- **Equivalent characterization of social opt:** (assuming $l^i$ is convex)
  \[
  u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p} l^e(f^e) - \sum_{e \in p} (l^e)'(f^e)f^e \leq 0, \quad \text{if } f^p_{k,j} = 0,
  \]
  \[
  = 0, \quad \text{if } f^p_{k,j} > 0.
  \]
- **ME price** = Marginal congestion cost + Monopoly markup
- For linear utility functions, path flows and flow rates of the ME and the social optimum are the same.
Performance of Monopoly Pricing

- Compare performance of monopoly pricing, WE(0), and social opt.

- **Example:** Consider the network below. Assume

  \[ u_A(x) = u_B(x) = 200x^\alpha, \quad 0 < \alpha \leq 1, \]
  \[ l(x) = x^\beta, \quad \beta \geq 1. \]

- \( U_{me}, U_{we}, U_{soc} \): total system utility at the ME, WE(0), and social optimum.
Exploiting Convexity of Latency Functions

- The more convex the latency function is, the better performance we have under monopoly pricing.
- Concavity in utility introduces distortion wrt to the social optimum.
Inelastic Traffic - Routing (with participation control)

- The utility function $u_j$ is a step function.

- With this utility function, decisions of user $j$ will be binary: either send $t_j$ units of traffic or do not send anything.
  - Routing with participation control

- Reasonable model of routing in the presence of service providers
  - Otherwise, the monopolist will set the prices equal to $\infty$.

- Includes implicit admission control.
Analysis of Inelastic Traffic

- The $u_j$ are no longer concave or continuous, therefore calculus-based analysis with fixed point theorems does not hold.

- WE can be defined equivalently in terms of flow variables and binary participation variables:

  $$(f_j^*, z_j^*) \in \arg \max_{f_j^p \geq 0, \ z_j \in \{0, 1\}} \left\{ z_j t_j - \sum_p (l^p(f^p) + q^p) f^p \right\}$$

  subject to:

  $$\sum_p f_j^p = t_j, \text{ if } z_j = 1$$

- In view of the Wardrop assumption, this converts the problem into a mixed integer-linear program and yields an equivalent characterization of a WE:

  - Positive flows on minimum effective cost paths
  - If $z_j = 0 \rightarrow f_j^p = 0, \ \forall \ p$; if $z_j = 1 \rightarrow \sum_p f_j^p = t_j$.
  - If $\min_p \{l^p(f^p) + q^p\} < 1 \rightarrow z_j = 1, \ \forall \ j$. 
Example

• There does not exist a WE at all price vectors $p$.

  Example: $A$ and $B$ inelastic with $t_A=1$, $t_B=1.5$

  ![Diagram]

• At price 0, there is no WE.

• At price 0.5, there is a WE in which $A$ sends his flow, $B$ does not.

• This is indeed the profit maximizing price.

• Consider the social problem for this example:

  $$\max_{x_A, x_B} \{u_A(x_A) + u_B(x_B) - l(x_A + x_B)(x_A + x_B)\}$$

  – At the social optimum, $A$ sends his flow, $B$ does not.
Our results

• Consider a general topology network
• Assume that $l^e$ is continuous and strictly increasing.
  – For a given price $q \geq 0$, if there exists a WE, it is unique.
• There exists a profit maximizing price at which there is a WE:
  – There exists a monopoly equilibrium.
• The flow allocation at the ME is identical to the social optimum.
• Entire user surplus extracted (special feature of monopoly).
• One interesting question is to look at the multiple provider case, where user surplus is positive.
Braess Paradox

- **Idea:** Addition of an intuitively helpful link negatively impacts users of the network

\[
\begin{align*}
\text{C}_{\text{eq}} &= \frac{1}{2} (\frac{1}{2}+1) + \frac{1}{2} (\frac{1}{2}+1) = \frac{3}{2} \\
\text{C}_{\text{sys}} &= 3/2
\end{align*}
\]

- Introduced in transportation networks [Braess 68], [Dafermos, Nagurney 84]
  - Studied in the context of communication networks, distributed computing, queueing networks [Altman et al, 03]

- Motivated research in methods of upgrading networks without degrading network performance
  - Leads to limited methods under various assumptions.
Generalized Braess Paradox

- **No prices:** Addition/deletion of a link one form of traffic restriction
  - [Hagstrom, Abrams 01] Let $f$ be a WE. A Braess paradox occurs if $\exists$ another flow distribution ("Braess distribution") $\tilde{f}$ st
    \[ l^p(\tilde{f}) \leq l^p(f), \quad \forall p, \]
    \[ l^{p'}(\tilde{f}) < l^{p'}(f), \quad \text{for some } p', \]
    where $l^p(f)$ : latency cost of path $p$ under flow $f$.
  - If WE is a social optimum, then there is no Braess paradox.
  - Braess distribution has lower total cost than WE.

- **With prices:**
  - Both remarks are not true [WE does not equalize latency costs].
  - Above condition need not always constitute a paradoxical situation when you consider flows switching from one path to another.
  - Need a new definition of Braess paradox.
Braess Paradox with Prices

- Given a price $q$, let $f$ be a WE, and $l(f) = [l^1(f), \cdots, l^{|P|}(f)]$ be the path latency vector.

- **Strong Braess Paradox:** A BP occurs if $\exists$ some other distribution of flows, $\bar{f}$, and a transformation $\Delta$ such that $\Delta \cdot f = \bar{f}$, $\Gamma_{k,j} = \bar{\Gamma}_{k,j}$, $l^j(f) \geq l^i(\bar{f})$, if $\Delta_{i,j} \neq 0$ (with strict inequality for some $i, j$), where $\Delta_{i,j}$ is the $(i, j)$ entry of matrix $\Delta$.

- **Remark:** $\Delta_{i,j} f^j$: amount of flow moved from path $j$ to path $i$.

- **Weak Braess Paradox:** A BP occurs if $\exists$ some other distribution of flows, $\bar{f}$, and a transformation $\Delta$ such that $\Delta \cdot f = \bar{f}$, $\Gamma_{k,j} = \bar{\Gamma}_{k,j}$, $l^p(f) \geq \Delta'_p \cdot l(\bar{f})$, $\forall p$, (with strict inequality for some $\tilde{p}$), where $\Delta_p$ is the $p$th column of $\Delta$.

- **Remarks:**
  - $\Delta_p f^p$: vector of redistribution of flow on path $p$.
  - $\Delta'_p l(\bar{f})$: avg latency seen by redist. of flow on path $p$ under $\bar{f}$. 
Strong and Weak Braess Paradox

WE: \( l^{\{a,c\}} = 19; \ l^{\{b,d\}} = 10.75; \ l^{\{a,e,d\}} = 15.25 \),

Bd: \( l^{\{a,c\}} = 18.75; \ l^{\{b,d\}} = 10; \ l^{\{a,e,d\}} = 10.25 \)

\[
\begin{align*}
18.75 & < 19; \ 10 < 10.75 \\
0.5 \times 18.75 + 0.5 \times 10 & = 14.375 < 15.2
\end{align*}
\]

- Weak Braess paradox occurs, Strong Braess paradox does not occur.
Monopoly Pricing and Braess Paradox

• Checking whether Strong/Weak Braess paradox does not occur is hard (need to consider all possible redistributions of flows)

• **Proposition:** Weak Braess paradox does not occur under monopoly prices.
  → Strong Braess paradox does not occur under monopoly prices.

• **Intuition:**
  – The sp, by setting profit maximizing prices, extracts the user surplus.
  – If there were a Braess distribution (WE not pareto optimal), the sp could extract the additional surplus and make more profit.
Extensions

• Efficient computational methods for ME.
• Multiple provider case $\rightarrow$ interesting efficiency results for the case of competition in congested markets.
• Different routing paradigms (selfish routing vs decentralized routing for a systemwide objective).
• Pricing for differentiated services.