Handout: Formal Theory of "The Firm as a System"

Notation:

$\mathbf{x} = (x_1,..,x_n)$ Firm choices; "endogenous variables."

These may be discrete or continuous; they may describe qualitative choices.

Examples of x_i: Policies, commission rates, technology adoption, supplier relations.

$\mathbf{t} = (t_1,..,t_n)$ Features of the firm's environment; "exogenous parameters."

Examples: Government regulation, institutional environment, prices.

Definition: Given two vectors, $\mathbf{y}$ and $\mathbf{z}$, we say that $\mathbf{y}\geq\mathbf{z}$ if $y_i \geq z_i$ for all i. That is, $\mathbf{y}$ is greater than $\mathbf{z}$ component-by-component.

Definition of ***Increasing Differences****:* There are four definitions which are equivalent for differentiable functions. Consider a function $f(x_1,x_2,\mathbf{y})$.

(i) Suppose that $\frac{\partial^2}{\partial x_1 \partial x_2} f(x_1,x_2,\mathbf{y})$ is well defined. Then f satisfies *increasing differences* in (x_1,x_2) if, for all x_1,x_2, and $\mathbf{y}$, $\frac{\partial^2}{\partial x_1 \partial x_2} f(x_1,x_2,\mathbf{y})\geq 0$.

(ii) Suppose that $\frac{\partial}{\partial x_1} f(x_1,x_2,\mathbf{y})$ is well defined. Then f satisfies *increasing differences* in (x_1,x_2) if, for all x_1,x_2, and $\mathbf{y}$, $\frac{\partial}{\partial x_1} f(x_1,x_2,\mathbf{y})$ is nondecreasing in x_2.

(iii) The function f satisfies *increasing differences* in (x_1,x_2) if, for all $x_1^H > x_1^L$, x_2, and $\mathbf{y}$, $f(x_1^H,x_2,\mathbf{y})-f(x_1^L,x_2,\mathbf{y})$ is nondecreasing in x_2.

(iv) The function f satisfies *increasing differences* in (x_1,x_2) if, for all $x_1^H > x_1^L$, $x_2^H > x_2^L$, and $\mathbf{y}$, $f(x_1^H,x_2^L,\mathbf{y})-f(x_1^L,x_2^L,\mathbf{y})\leq f(x_1^H,x_2^H,\mathbf{y})-f(x_1^L,x_2^H,\mathbf{y})$.

Example: Suppose that $(x_1,x_2) \in \{0,1\}\times\{0,1\}$. Let x_1 be training and x_2 be purchase of computers. Then $f(x_1,x_2)$ satisfies increasing differences if $f(1,0)-f(0,0)\leq f(1,1)-f(0,1)$. The interpretation is that the LHS is the incremental returns to training when the firm doesn't buy computers, while the RHS is the incremental returns to training when the firm does buy computers.

Definition of (Edgeworth) ***complements***: x_1 and x_2 are complements to a firm if the firm's profit function satisfies increasing differences in (x_1,x_2).

Example continued: Thus, we say that training and computer purchases are complements if buying computers increases the incremental returns to training.

For your information: The Milgrom/Roberts article uses another term in the technical section. This term is *supermodular*. A function $f(\mathbf{x})$ is called supermodular in $\mathbf{x}$ if it satisfies increasing differences in (x_i,x_j) for all $i\neq j$. In other words, each pair of variables is complementary.

"Increasing Differences Comparative Statics Theorem." (Topkis, 1978). Consider a vector of choices, $\mathbf{x}=(x_1,..,x_n)$, chosen from a set $X_1\times\cdots\times X_n$. These sets X_i may be finite-valued (i.e. $\{0,1\}$). Further, consider a vector of exogenous parameters $\mathbf{t} = (t_1,..,t_m)$. The objective function is given by $\pi(\mathbf{x},\mathbf{t})$.

Condition 1: "Complementarities between each pair of choice variables."

> $\pi(\mathbf{x},\mathbf{t})$ satisfies Condition 1 if $\pi(\mathbf{x},\mathbf{t})$ satisfies increasing differences in (x_i,x_j) for all $i\neq j$. (Equivalently, if $\pi(\mathbf{x},\mathbf{t})$ is *supermodular* in $\mathbf{x}$).

Condition 2: "Exogenous variables increase the returns to the choice variables."

> $\pi(\mathbf{x},\mathbf{t})$ satisfies Condition 2 if $\pi(\mathbf{x},\mathbf{t})$ satisfies increasing differences in (x_i,t_k) for all $i=1,..,n$, all $k=1,..,m$.

Theorem: Let $\mathbf{x}^*(\mathbf{t}) = \arg\max_{\mathbf{x}}\pi(\mathbf{x},\mathbf{t})$. (Note: $\mathbf{x}^*(\mathbf{t})=(x_1^*(\mathbf{t}),..,x_n^*(\mathbf{t}))$). If $\pi(\mathbf{x},\mathbf{t})$ satisfies Conditions 1 and 2, then $\mathbf{x}^*(\mathbf{t})$ is nondecreasing in $\mathbf{t}$. That is, for $\mathbf{t}^H\geq\mathbf{t}^L$ (component by component), the optimal choices each satisfy $x_i^*(\mathbf{t}^H)\geq x_i^*(\mathbf{t}^L)$.

Notes about theorem: The theorem says that if we increase the returns to one or more choice variables, and further the choices are pairwise complementary, then the optimal choices will all increase (weakly). Condition 1 implies that the indirect effects of adjusting the choices will reinforce the direct effects of the changes in parameters. Thus, if training and computers are complements, then a decrease in the price of computers will lead the firm to increase *both* training and computers. It is interesting that only pairwise relationships must be specified to make this theorem go, and further the theorem does not rely on the standard First-Order-Condition kind of logic, since that logic might not even apply.

Example: Firm as an Incentive System.

Recall our multitask model. The principal maximizes the total certainty equivalent by choosing commission rates, β_1 and β_2. Let $\mathrm{TCE}(\beta_1,\beta_2;\sigma_1^2,\sigma_2^2)$ be this certainty equivalent. The commissions are the choice variables, and the variances of the measurement errors are exogenous parameters. Recalling that the agent's risk premium is part of this expression, we can (when the measurement errors are independently distributed) write the TCE as follows:

$$\mathrm{TCE}(\beta_1,\beta_2;\sigma_1{}^2,\sigma_2{}^2) = v_1\, e_1(\boldsymbol{\beta}) + v_2\, e_2(\boldsymbol{\beta}) - C(e_1(\boldsymbol{\beta}),e_2(\boldsymbol{\beta})) - \tfrac{1}{2} r\beta_1{}^2\sigma_1{}^2 - \tfrac{1}{2} r\beta_2{}^2\sigma_2{}^2$$

We discussed in class conditions under which $\mathrm{TCE}(\beta_1,\beta_2;\sigma_1{}^2,\sigma_2{}^2)$ satisfies increasing differences in (β_1,β_2), that is, β_1 and β_2 are complements. We won't analyze that here but just take it as given. Now let's think about what happens if $\sigma_2{}^2$ falls, that is, $(-\sigma_2{}^2)$ goes up (activity 2 becomes easier to measure). To apply the comparative statics theorem, we need to check Conditions 1 and 2. We are taking Condition 1 as given from our in-class analysis. Now consider Condition 2. Well, $\frac{\partial^2}{\partial\beta_1\partial(-\sigma_2^2)} TCE(\beta_1,\beta_2;\sigma_1^2,\sigma_2^2) = 0$, while $\frac{\partial^2}{\partial\beta_2\partial(-\sigma_2^2)} TCE(\beta_1,\beta_2;\sigma_1^2,\sigma_2^2) = r\beta_2 \geq 0$. So, Condition 2 is satisfied. We then conclude that by the theorem, making activity 2 easier to measure leads us to use higher commissions on both activity 1 and activity 2.

The intuition is that, when activity 2 is easier to measure, the increasing the commission does not increase the risk premium as much as before. Thus, the firm wants to give higher incentives on activity 2. But this will cause the agent to substitute effort away from activity 1, unless the firm also increases the commission rate on activity 1.