Comparative Statics Problems—Solutions

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1. f has increasing differences, so: $f(y'',\theta) - f(y',\theta)$ is non decreasing in $\theta \forall y'' > y'$. Let $x'' > x'$. g is increasing, so $y'' = g(x'') > g(x') = y'$. This means that $f(g(x''),\theta) - f(g(x'),\theta)$ is increasing in θ ,so h has increasing differences.

2. We are in the continuously differentiable world, so we only have to check that cross partials are positive:

$$\frac{\partial^2 f}{\partial x \partial \theta} = -g''(x-\theta) \geq 0 \Longleftrightarrow g \text{ concave}$$

To think about the non-differentiable world, we need to write down the definition of increasing differences:

$$g(x''-\theta'') - g(x'-\theta'') > g(x''-\theta') - g(x'-\theta'), \text{ for } x'' > x', \text{ and } \theta'' > \theta'$$

We are free to pick whatever we like for the xs and the θs. I'm going to choose $x'' = x+2h$, $x' = x+h$, $\theta'' = 2h$, $\theta' = h$. You can see that $x'' > x'$, and $\theta'' > \theta'$ for all x and for all $h > 0$. If we plug these into the definition of increasing differences of f, we find that:

$$g(x) - g(x-h) > g(x+h) - g(x), \text{or}$$

$$g(x) > \frac{1}{2}\left[g(x+h) + g(x-h)\right],$$

This, as was pointed out to me, does not imply that g is concave, unless g is continuous. Sorry if I gave you any other impression. You have to use a pretty bizzarre function to find a counter-example, but that's math. for you.
On the other hand, if g is concave, it always has increasing differences whether continuous or not (though concave everywhere is pretty damn close to continuity). The proof goes something like this: suppose you have $x'' > x'$ and $\theta'' > \theta'$. Then let $A = x' - \theta'$, $B = x' - \theta''$, $C = x'' - \theta''$, $D = x'' - \theta'$. Clearly, $A < B, C < D$. B and C are not rankable, but that won't matter. Assume that $B < C$. You can do the other case as well; it works just the same. The proof that g has increasing differences comes from looking at the concavity conditions at B between A and C, and at C between B and D. Let $B - A = \Delta = D - C$, and let $C - B = \Gamma$. Let $p = \Delta/(\Delta+\Gamma)$, and let $q = \Gamma/(\Delta+\Gamma)$.

$$g(B) > g(A) + p\left(g(C) - g(A)\right)$$

$$g(C) > g(B) + q\left(g(D) - g(B)\right)$$

Add these two inequalities up, and rearrange, and you get the increasing differences definition.

3. If x'' and x' are optimal, it must be that:

$$f(x'',\theta'') \geq f(x,\theta''), \text{ and}$$
$$f(x',\theta') \geq f(x,\theta') \forall x$$

Since we may choose x to be anything we like in the expression above,choose $x = x'$, so that $f(x'',\theta'') \geq f(x',\theta'')$, and $f(x',\theta') \geq f(x'',\theta')$. Add these two up:

$$f(x'',\theta'') + f(x',\theta') \geq f(x',\theta'') + f(x'',\theta')$$
$$\Longleftrightarrow$$
$$f(x'',\theta'') - f(x',\theta'') \geq f(x'',\theta') - f(x',\theta')$$

which could not be the case if $x'' < x'$, because then strictly increasing differences would imply the opposite. Therefore, $x'' \geq x'$ as required.

4. Let's take the ln of the utility function, and maximise that:

$$\max[\ln(p-c) + \ln Q(p,\theta)]$$

First consider changing θ. This is like the template problem $\max\left(f(x,\theta)+g(x)\right)$ with "f" replaced by $\ln Q$ and "g" replaced by $\ln(p-c)$. Increasing differences of $\ln Q$ is therefore sufficient to show that p^* is non-decreasing in θ. Here we may check cross partials:

$$\frac{\partial \ln Q}{\partial p} = \frac{1}{p}(\varepsilon(p)+\theta)$$
$$\frac{\partial^2 \ln Q}{\partial p \partial \theta} = \frac{1}{p} > 0$$

In the case where we are changing c, we can just flip the rôles of $\ln(p-c)$ and $\ln Q$. Then the cross partial of $\ln(p-c)$ needs to positive: it is.