## *Final Set of Problems: 14.121 Fall 1996*

1. [from 1996 waiver exam] Consider a profit-maximizing fast-food restaurant which invests money in equipment ($k$), at a rental price of capital given by $r$, and hires workers ($l$). The cost *per hour* from hiring $l$ hours of labor is $W(l)$, which is nondecreasing in $l$. The firm produces fast food according to the function $F(k,l)$, and sells this output at a constant price $p$. The equipment $k$ is fixed in the short run. Suppose that we are currently in a long-run equilibrium, and the restaurant is currently using $l_0$ labor at a cost of $W(l_0)$ = \$4/hr.

   Assume throughout the problem that (i) $F$ is nondecreasing, and (ii) there is a unique, interior optimal choice of equipment and labor for all relevant parameter values. If you need *any* additional assumptions, you must state them.

Now suppose that Congress votes to institute a minimum wage of $w^{\min}$=\$5.

(i) Consider the firm's short run problem of choosing a new level of labor, $l_s$. Can you conclude that $l_s \leq l_0$? [Your analysis here should be partial equilibrium; do not worry about general equilibrium feedbacks in the labor supply. The hard part is figuring out the new cost of labor function (with the minimum wage) looks like. It might to draw a graph of wages against labor.]

Now suppose that the firm has been producing in a long-run equilibrium with the minimum wage in place, using labor $l_0'$ at wage $w^{\min}$=\$5. Then Congress raises the minimum wage again, to \$5.25.

(ii) Can you conclude that the firm will adjust its demand for labor more in the long run than in the short run? That is, let the firm's new short run labor labor choice be $l_s'$, and the new long-run labor choice be $L$. Then, under what conditions can we conclude that $L \leq l_s' \leq l_0'$?

(iii) Finally, let us change the problem a little bit. Forget all of our previous assumptions about the cost of labor, and suppose instead that the firm's *total* cost (not per hour cost) of hiring $l$ hours of labor is given by a function $h(l;t)$. Further, suppose that the marginal cost of labor ($h_l(l;t)$) is nondecreasing in $t$, but you make no further assumptions about $h$. In response to an increase in $t$, what happens to the firm's short run choice of $l$? Compare your answer to part (i), and explain similarities and differences. How does you answer to part (ii) change if we consider the response to a discrete change in $t$ instead of the change in the minimum wage?

2. Suppose that a firm's production depends on two inputs, $X$ and $Y$, which are both chosen from the set $\{L,M,H\}$. Suppose that the firm's production as a function of those two choices, $f(X,Y)$, are given by the entries in the following table:

| | | $Y$ | | |
|---|---|---|---|---|
| | | L | M | H |
| $X$ | H | 7 | 12 | 14 |
| | M | 4 | 7 | 11 |
| | L | 2 | 4 | 6 |

The cost of $X$ is $g(X)$ and the cost of $Y$ is $h(Y)$. Does the firm's profit function satisfy increasing differences in $(X,Y)$? Is there one element of the table you could change to ensure that increasing differences holds?

3. Consider the following profit function (which is differentiable as well as concave in $(x,y)$):

$$\pi(x,y|t) = tx + \sqrt{t}\,y - \frac{x^2}{2} - \frac{y^2}{2} - \frac{1}{3}xy$$

(a) **Suppose that *y* is fixed.** The firm chooses $x$ to maximize profits. Is the optimal choice of $x$ is increasing or decreasing in $t$?
(b) **Suppose that *x* is fixed.** The firm chooses $y$ to maximize profits. Is the optimal choice of $y$ is increasing or decreasing in $t$?
(c) **Now, suppose that the firm chooses both *x* and *y*.** Solve for the optimal choices of $x$ and $y$ as a function of $t$ when both $x$ and $y$ are choice variables. Calculate the derivatives of $x^*$ and $y^*$ with respect to $t$. At $t$=1, do the optimal choices of $x$ and $y$ increase or decrease as a result of a small increase in $t$? What about at $t$=4?
(d) Interpret your answers in the context of the monotone comparative statics theorems.

4. A firm assigns decision-making authority to its central office and a division manager depending on what the "state of the world," $S$, is. (I.e., if lots of workers are sick today, if there's an unexpected increase in demand for all divisions). States are described by $S\in[0,1]$, and the probability that the state $S$ occurs is represented by the density function $f(s)$. The firm's choice of *discretion* is represented by $x\in[0,1]$, so that the division manager makes the decision for $S < x$ and the central office makes the decision otherwise. The firm also chooses how much *training*, $t$, to give the division manager, and how much *administrative support*, $a$, to give the central office. When the division manager makes the decision in state $S$, the firm receives payoff $M(t,S)$, while when the central office makes the decision, the firm receives payoff $C(a,S)$. Training costs $P_t$ per unit, while administrative support costs $P_a$ per unit. All fixed costs or benefits of choosing discretion level $x$ are included in the term $K(x)$. Thus, the firm's expected profit function is given as follows:

$$\pi(x,t,a|P_t,P_a) = \int_0^x M(t,S)f(S)dS + \int_x^1 C(a,S)f(S)dS - P_a\cdot a - P_t\cdot t - K(x)$$

Assume that $M$ and $C$ are increasing in $t$ and $a$. No assumptions are made about $K$.

(a) Predict how the optimal choices of $x$, $t$, and $a$ change when the price of training changes. Repeat your analysis for the price of administrative support.
(b) *Optional:* Suppose that the prices faced by firms for these choices are independently distributed in the population of firms, and that an economist measures the pairwise correlations between $(x,t,a)$ in that population. Will these correlations be positive or negative?

5. Are the following cost functions valid solutions to our standard cost-minimization problem?

(a) $C(q;r,w) = q\cdot(r+w^2)$
(b) $C(q;r,w) = q\cdot(r-w)$
(c) $C(q;r,w) = q\ r\cdot\exp\{-w/r\}$
(d) $C(q;r,w) = q\cdot\sqrt{r^2+w^2}$

6. A firm has a production function given by $f(k,l) = k^{\alpha} \cdot l^{\beta}$, where $\alpha > 0$, $\beta > 0$, and $\alpha + \beta < 1$. Input prices are $r$ and $w$, and the output price is $p$.
   (a) Derive the firm's maximum profits $\pi^*(p,r,w)$ and the unconditional factor demand functions, $k(p,r,w)$ and $l(p,r,w)$, which solve the firm's profit maximization problem.
   (b) What happens if $\alpha + \beta > 1$? $\alpha + \beta = 1$?