Microeconomic Theory I *14.121* *Fall 1996*

Problem Set I: Solutions

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1. The axioms are, to summarise:
   - **A 1** R is a complete ordering on X, the space of commodity bundles.
   - **A 2** R is weakly monotonic, or $(x \geq y) \Rightarrow (x \, R \, y)$.
   - **A 3** Local non-satiation: $\forall x \in X, \delta>0, \exists y \in X: (|y - x| < \delta) \wedge (y \, P \, x)$
   - **A 4** Continuity: $\forall x \in X$, $S^{+}(x)$ and $S^{-}(x)$ are closed in X.
   - **A 5** Strict convexity of preferences.

   P and I are defined in terms of R:
   $(x \, I \, y) \Leftrightarrow (x \, R \, y) \wedge (y \, R \, x)$
   $(x \, P \, y) \Leftrightarrow (x \, R \, y) \wedge \neg (y \, R \, x)$

   (a) $\forall x, (x \, R \, x)$, by **A1**, so $\forall x, (x \, R \, x) \wedge (x \, R \, x)$, which if you look at the definition of I, says the same as x I x

   (b) $(x \, P \, y) \wedge (y \, P \, z) \Leftrightarrow (x \, R \, y) \wedge (y \, R \, z) \wedge \neg(y \, R \, x) \wedge \neg (z \, R \, y)$ from the definition of P.
   $\Rightarrow (x \, R \, z) \wedge \neg(y \, R \, x) \wedge \neg (z \, R \, y)$, by transitivity of R (**A1**). Now by completeness of R, $\neg(y \, R \, x) \Leftrightarrow (x \, R \, y)$. It's also true that $\neg(y \, R \, x) \wedge \neg (z \, R \, y) \Rightarrow \neg((y \, R \, x) \wedge (z \, R \, y))$
   $(x \, P \, y) \wedge (y \, P \, z) \Rightarrow (x \, R \, z) \wedge \neg ((y \, R \, x) \wedge (z \, R \, y))$, so when we use transitivity again,
   $\Rightarrow (x \, R \, z) \wedge \neg (z \, R \, x) \Leftrightarrow (x \, P \, z)$ QED.
   The others are almost exactly the same as this one: I'll do them in class

   (c) $(x \, I \, y) \Leftrightarrow (x \, R \, y) \wedge (y \, R \, x) \Leftrightarrow (u(x) \geq u(y)) \wedge (u(y) \geq u(x)) \Leftrightarrow (u(x) = u(y))$
   $(x \, P \, y) \Leftrightarrow (x \, R \, y) \wedge \neg (y \, R \, x) \Leftrightarrow (u(x) \geq u(y)) \wedge \neg (u(y) \geq u(x)) \Leftrightarrow (u(x) > u(y))$

   (d) Something is a utility function if it is $u: X \rightarrow \mathbf{R}$ and $u(x) = u(y)$ iff x I y, and $u(x) > u(y)$ iff x P y. We know that u is a utility function, and v is a strictly monotonic transformation of u. Obviously $v: X \rightarrow \mathbf{R}$. Now $F(a) > F(b)$ iff $a > b$ so $v(x) > v(y)$ iff $F(u(x)) > F(u(y))$ iff $u(x) > u(y)$, and since u is a utility function, iff x P y. Exactly the same way, $F(a) = F(b)$ iff $a = b$, and the logic works analogously.

2.

   (a) I'm going to define the function $V(b,\mathbf{x}) = U(b + x_1, \ldots, x_n)$, just to be clearer about what is being differentiated where. V is is the utility function. The marginal utility of good 1 is $MU_1(b, \mathbf{x}) = \partial/\partial x_1 V(b, \mathbf{x}) = U_1(b + x_1, \ldots)$. If we change b, then the effect this has on MU, keeping everything else constant is $\partial/\partial b \, U_1(b + x_1, \ldots) = U_{11}(b + x_1, \ldots)$. So asking if changing b increases or decreases MU of good one is the same as asking if good one exhibits decreasing or increasing marginal utility. This is not determined, and is not invariant under general monotonic transformations.

(b) We will be minimising **p.x** subject to the constraint that $V(b, \mathbf{x}) = \nu$. The Lagrangian will be $\Lambda = \mathbf{p.x} - \lambda(\nu - V(b, \mathbf{x}))$. What is the derivative of expediture wrt b? Well, we can use the envelope theorem: Expenditure depends on prices, utility, and other parameters (b). The derivative of $E(b, \nu, \mathbf{p})$ with respect to b is just the partial derivative of the lagrangian wrt b: $\partial E(b, \nu, \mathbf{p})/\partial b = \lambda \partial V/\partial b = \lambda U_1(b + x_1, ...)$. The FOCs for the minimisation problem are: $p_i = -\lambda V_i(b, \mathbf{x})$. In particular, the FOC for x_1 is just $p_1 = -\lambda U_1(b + x_1, ...)$. So, $\partial E(b, \nu, \mathbf{p})/\partial b = -p_1$. That's as directly observable as you get.

(c) You know that if you differentiate the expenditure function with respect to a price, you get the quantity demanded (the Hicksian demand function). So $\partial E/\partial p_i = x_i$. Of we differentiate this with respect to b, $\partial x_i/\partial b = \partial^2 E/\partial p_i \partial b = \partial/\partial p_i \, (\partial E/\partial b)$, since partial derivative operators commute, and this is $= \partial/\partial p_i \, (-p_1)$. So the answer is 0 for $i > 1$, and -1 for $i = 1$. Simple as that.

3. We have to check that WARP isn't violated, and that SARP (a.k.a. GARP) isn't violated either. Since you didn't know about SARP when you did this problem set, I didn't expect you to check for this. That means that for all the bundles given, we have to see if any are revealed preferred to any of the others. I shall call the bundles of goods a, b, c, and d. I shall write "a is revealed preferred to b" as $a >_{rp} b$, and I shan't be too careful about strict vs. non-strict revealed preferences.

First to check: is a revealed preferred to anything? In other words, could the consumer have afforded any of the other bundles at the prices he or she bought a? To buy b at a's prices (1,1,1) would cost 14. To buy c would cost 19, to buy d would cost 19. Thus b,c, and d were all affordable when the consumer chose a. Hence $a >_{rp} b$, $a >_{rp} c$, and $a >_{rp} d$.

Now we have to do the same for b, c, and d. b first. At b's prices (3,1,1), a costs 40, c costs 45, and d costs 49. b is not revealed preferred to anything. At c's prices (1,2,2) a costs 30, b costs 25, and d costs 23. Thus the consumer could have afforded b or d, but chose c: $c >_{rp} b$ and $c >_{rp} d$. Finally, the same for d. at d's prices (1,1,2), a cost 25, b cost 20, and c cost 22. Only b was affordable, so $d >_{rp} b$.

Stare for a while at the following table:

$a >_{rp} b$	$a >_{rp} c$	$a >_{rp} d$
$c >_{rp} b$		$c >_{rp} d$
$d >_{rp} b$		

These are the only revealed preferences we may conclude from the four bundles chosen. To check WARP, if it's violated, we have to find two bundles such that $x >_{rp} y$ and $y >_{rp} x$. There are none, so WARP is OK.

SARP is a little trickier. For it to hold, any time we have $x > y$ and $y > z$, we also have to have NOT($x < z$). $a >_{rp} c$ and $c >_{rp} b$ and $c >_{rp} d$. Since $a >_{rp} c$, and $a >_{rp} d$ this is OK. $c >_{rp} d$ and $d >_{rp} b$. Since $c >_{rp} b$, SARP is OK. Thus these revealed preferences are consistent with a utility maximising non-satiable consumer.

4.

(a) The Lagrangian is $L = U(\mathbf{x}) - \lambda(\mathbf{p.x} - Y)$. The first order conditions are: $U_1 = \lambda p_1$ and $U_2 = \lambda p_2$. When you differentiate the utility function properly you get:

$$\lambda p_1 = \alpha_1 \rho x_1^{\rho-1} \Phi^{\frac{1}{\rho}-1}$$

$$\lambda p_2 = \alpha_2 \rho x_2^{\rho-1} \Phi^{\frac{1}{\rho}-1}$$

Where I have defined $\Phi = \alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}$ to make the algebra a little easier.

(b) I'm afraid this means that you have to solve for x_1, x_2, and utility in terms of income, prices and other parameters. This is a little messy. If you multiply the first order conditions by x_1 and x_2 respectively, and then add them up, the budget constraint tells you that:

$$\lambda = \frac{\rho \Phi^{1/\rho}}{Y}.$$

Substituting this back into the first order conditions gives you equations without λ:

$$p_1 / Y = \alpha_1 x_1^{\rho-1} \Phi^{-1}$$

$$p_2 / Y = \alpha_2 x_2^{\rho-1} \Phi^{-1}.$$

The rest is just mathematical grind. Rearrange these equations:

$$x_1^{\rho} = \left(\frac{Y \alpha_1}{\Phi p_1} \right)^{\frac{\rho}{1-\rho}}$$

$$x_2^{\rho} = \left(\frac{Y \alpha_2}{\Phi p_2} \right)^{\frac{\rho}{1-\rho}}$$

Multiply the first by α_1, the second by α_2 and add them up. The left hand side is Φ. Thus:

$$\Phi = \left(\frac{Y}{p_1 p_2} \right)^{\rho} \left\{ \alpha_1^{\frac{1}{1-\rho}} p_2^{\frac{\rho}{1-\rho}} + \alpha_2^{\frac{1}{1-\rho}} p_1^{\frac{\rho}{1-\rho}} \right\}^{1-\rho}.$$

But that gives us the indirect utility function, as $u = \Phi^{1/\rho}$. If I define $\sigma = \rho/(1-\rho)$, then we get:

$$v(p_1, p_2, Y) = Y \left(\alpha_1^{1+\sigma} p_1^{-\sigma} + \alpha_2^{1+\sigma} p_2^{-\sigma} \right)^{1/\sigma}.$$

You substitute this back into the equations for x_1 and x_2 in terms of Φ to get the Marshallian demand functions:

$$D^1 = Y \left(\frac{\alpha_1}{p_1} \right)^{\sigma+1} \left[\alpha_1^{1+\sigma} p_1^{-\sigma} + \alpha_2^{1+\sigma} p_2^{-\sigma} \right]^{-1}, \text{ and similar for } D^2.$$

c) Before rushing off any further, I'm going to define some quantities which will make the algebra tolerable. Let $q_1 = p_1/\alpha_1$ and similarly for q_2. Armed with this,

$$D^1 = Yq_1^{-\sigma-1}Z^{-1}$$

$$D^2 = Yq_2^{-\sigma-1}Z^{-1}$$

$$V = YZ^{1/\sigma}$$

$$Z = \alpha_1 q_1^{-\sigma} + \alpha_2 q_2^{-\sigma}$$

You can see that D^i are homogeneous of degree zero in ($\boldsymbol{p}$, Y), so you know that the first adding up theorem must hold. It's then just a question of believing in your calculus:

$$p_1 \frac{\partial D^1}{\partial p_1} = q_1 \frac{\partial D^1}{\partial q_1} = q_1^{-\sigma-1}YZ^{-1}\left\{-(\sigma+1) + \sigma\alpha_1 Z^{-1}q_1^{-\sigma}\right\}$$

$$p_2 \frac{\partial D^1}{\partial p_2} = q_2 \frac{\partial D^1}{\partial q_2} = q_1^{-\sigma-1}YZ^{-1}\left\{\sigma\alpha_2 Z^{-1}q_2^{-\sigma}\right\}$$

$$Y \frac{\partial D^1}{\partial Y} = q_1^{-\sigma-1}YZ^{-1}$$

add them up and you get zero. The one for D^2 is going to work by symmetry.
For AUT 2,

$$p_1 \frac{\partial D^1}{\partial p_1} = q_1 \frac{\partial D^1}{\partial q_1} = q_1^{-\sigma-1}YZ^{-1}\left\{-(\sigma+1) + \sigma\alpha_1 Z^{-1}q_1^{-\sigma}\right\}$$

$$p_2 \frac{\partial D^2}{\partial p_1} = q_2 \frac{\alpha_2}{\alpha_1}\frac{\partial D^2}{\partial q_1} = q_1^{-\sigma-1}YZ^{-1}\left\{\sigma\alpha_2 Z^{-1}q_2^{-\sigma}\right\}$$

$$D^1 = q_1^{-\sigma-1}YZ^{-1}$$

Add them up and you get zero.
For adding up theorem 3,

$$p_1 \frac{\partial D^1}{\partial Y} + p_2 \frac{\partial D^2}{\partial Y} = \alpha_1 q_1 \frac{\partial D^1}{\partial Y} + \alpha_2 q_2 \frac{\partial D^2}{\partial Y} = Z^{-1}\left\{\alpha_1 q_1^{-\sigma} + \alpha_2 q_2^{-\sigma}\right\} = 1$$

d) The Lagrangian is $L = \boldsymbol{p}.\boldsymbol{x} - \lambda(u(\boldsymbol{x})-u)$. The first order conditions are:

$$p_i = \lambda u_i(\mathbf{x})$$

$u(\mathbf{x}) = u$

These are the same as for the UMP in (a). No surprise there. Knowing that, you don't have to solve the equations all over again. Use duality for e)

e) Since $u = V(\boldsymbol{p},E(\boldsymbol{p},u))$, it is easy to see that:

$$E = uZ^{-1/\sigma}.$$

Now for the Hicksian demand functions:

$$h^1 = \frac{\partial E}{\partial p_1} = \frac{1}{\alpha_1}\frac{\partial E}{\partial q_1} = uZ^{-1-1/\sigma}q_1^{-\sigma-1}$$

$$h^2 = \frac{\partial E}{\partial p_2} = \frac{1}{\alpha_2}\frac{\partial E}{\partial q_2} = uZ^{-1-1/\sigma}q_2^{-\sigma-1}$$

f) Ah. Slutsy's Theorem. I'm only going to show you the (1,2) part of it. The others work in a very similar way.

$$\frac{\partial D^1}{\partial p_2}+\frac{\partial D^1}{\partial Y}D^2 = Yq_1^{-1-\sigma}q_2^{-1-\sigma}Z^{-2}(1+\sigma).$$

Now, you use duality, that $Y = uZ^{-1/\sigma}$.

$$\frac{\partial D^1}{\partial p_2}+\frac{\partial D^1}{\partial Y}D^2 = uq_1^{-1-\sigma}q_2^{-1-\sigma}Z^{-2-1/\sigma}(1+\sigma).$$

Differentiate the hicksian demand with respect to price:

$$\frac{\partial h^1}{\partial p^2}=\frac{1}{\alpha_2}\frac{\partial h^1}{\partial q^2}=uq_1^{-1-\sigma}q_2^{-1-\sigma}Z^{-2-1/\sigma}(1+\sigma).$$

This is the same as the Marshallian expression in the line before: Slutsky.

g) This means that we have to find the substitution matrix:

$$\mathbf{S}=(1+\sigma)uq_1^{-\sigma-1}q_2^{-\sigma-1}Z^{-2-1/\sigma}\begin{pmatrix}-\frac{q_2\alpha_2}{q_1\alpha_1} & 1\\ 1 & -\frac{q_1\alpha_1}{q_2\alpha_2}\end{pmatrix}$$

It's symmetric. Own substitution terms are negative (the diagonal elements). The determinant is zero, but the matrix isn't zero. This implies that rank($\mathbf{S}$) = 1. It is, thus, negative semi-definite.

5. a) The budget constraint has to be satisfied. So you can solve $\boldsymbol{p}.\boldsymbol{x} = Y$ to get x_3.

b) Yes, they are homogeneous of degree zero in ($\boldsymbol{p}$, Y).

c) a) and b) tell you that the AUTs will be satisfied automatically, and will give you no restrictions on the constants. The only thing which might is the substitution matrix. Of that, only the (1,2) block will say anything. $\mathbf{S}$ has to be symmetric. Thus, $S_{12} = S_{21}$ Using Slutsky to calculate this from the Marshallian demands,

$$\frac{\partial x_1}{\partial p_2}+\frac{\partial x_1}{\partial Y}x_2 = \frac{\partial x_2}{\partial p_1}+\frac{\partial x_2}{\partial Y}x_1$$

$$\Leftrightarrow$$

$$\frac{\beta}{p_3}+\frac{\delta}{p_3}\left(\alpha+\beta\frac{p_1}{p_3}+\gamma\frac{p_2}{p_3}+\delta\frac{w}{p_3}\right)=\frac{\beta}{p_3}+\frac{\delta}{p_3}\left(100-5\frac{p_1}{p_3}+\beta\frac{p_2}{p_3}+\delta\frac{w}{p_3}\right)$$

Note that this has to hold for all positive prices and wealth levels. All the coefficients of the various terms must be the same, so that $\alpha = 100$, $\beta = -5$, and $\gamma = \beta = -5$, unless $\delta = 0$. This question is designed so that δ is not 0. Unfortunately, this doesn't work. What other information can we use? Own substitution terms must be negative. $S_{11} = S_{22}$ trivially, so you only need to check one of them:

$$S_{11}=\frac{1}{p_3}\left\{-5+\delta\left[100-5\frac{p_1+p_2}{p_3}\right]+\delta^2\frac{w}{p_3}\right\}\le 0 \quad \forall p_i, w$$

This means that δ has to be zero. Why? S_{11} is quadratic in δ, with positive coefficient

of δ^2, w/p_3. Thus, if we increase w enough, S_{11} will become positive, unless $\delta = 0$. If δ=0, of course, we can say nothing about the other parameters. If we restricted the demand equations to be valid only for a certain range of prices and wealth, we could have δ non-zero, but as stated, the question leads you to assume that the demand equations hold for all wealths and prices. The question screwed up. Sorry.

d) For any prices or wealth level, $x_1 = x_2$. This means that, for constant x_3, the indifference curves must be L-shaped. I'll draw appropriate diagrams in class. This means that the utility function restricted to the hyperplane x_3 = constant must be a multiple of the function $\max[x_1, x_2]$.

e) $u(\boldsymbol{x}) = \phi(\max[x_1, x_2], x_3)$, Where ϕ: $\mathbf{R}^2 \rightarrow \mathbf{R}$ will be quasi-concave, I suspect.