Final Set of Problems—Solutions

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It will be helpful in most of this to recall the MCS Theorems:

1V–MCST $\arg\max f(x,\theta)+g(x)$ is non-decreasing in θ for all g if f has increasing differences.

MV–MCST $\arg\max f(\mathbf{x})+\sum_n x_n\theta_n$ is non-decreasing in θ_m for all other θ if f is supermodular (which means that it has increasing differences in all pairs of variables).

1. Susan discussed this quite a lot in class today. I'll draw graphs for you this Thursday.

   (a) You can't conclude that $l_s \leq l_0$. That's because the minimum wage may actually reduce the effective marginal cost of employing more l by removing the effects of having to offer all workers an increase.

   (b) The problem the firm faces is
   $$\max_{k,l} pF(k,l) - wl - rk$$
   We wish to consider the effects of an increase in w^{min}. The optimal long-run k and l are non-decreasing in $-w$ if F is supermodular, *i.e.* if $F_{kl} \geq 0$. If $F_{kl} \leq 0$, we can get the same result by noting that $F(-k,l)$ is supermodular. Thus in both these cases $L \leq l_0'$. How is l_s' related to L and l_0'? let $\hat{l}(k,w)$ be the optimal level of labour, given fixed wage and capital.
   First the case where $F_{kl} \geq 0$. Then the long-run capital chosen, $K \leq k_0'$, the level of capital at the old wage. $\hat{l}$ is non-decreasing in k, so $l_s' \geq L$. But $\hat{l}$ is non-increasing in w, so $l_s' \leq l_0'$. Thus $L \leq l_s' \leq l_0'$, and the long run change in labour is bigger than the short run change. Capital and labour are global complements, and so a rise in wage induces a fall in short term labour, induces a desired fall in long term capital and labour.
   Next, consider where capital and labour are global substitutes, $F_{kl} \leq 0$. Now the long-run choice of capital is non-decreasing in the wage, so $K \geq k_0'$. But then, $\hat{l}$ is decreasing in capital, so we still have the required result.
   Infinitesimally, the result works anyway. If the sign of F_{kl} is ambiguous, then you can't tell what happens.

   (c) Here, $-h(l,-t)$ has increasing differences, so an increase in t always reduces l. In part i), increasing the minimum wage didn't always increase or decrease the marginal employment cost, so this result doesn't apply.

2. You just have to check this by brute force. We want $\pi(H,Y)-\pi(M,Y)$ to be increasing in Y. It isn't. You need to increase $\pi(H,H)$ by at least 2. The other ones all work just fine.

3. For parts a) and b) it's easy. Both π_{ty} and π_{tx} are positive, so that x and y are increasing as functions of t when the other one is fixed. To solve both for optimal x and y, write down the first order conditions:
   $$t - x - \frac{1}{3}y = 0$$
   $$t^{1/2} - y - \frac{1}{3}x = 0$$
   Solve them and find that
   $$x^* = \frac{1}{8}(9t - 3t^{1/2})$$
   $$y^* = \frac{3}{8}(3t^{1/2} - t)$$
   The derivatives of the optimal x and y with respect to t are:
   $$x^{*\prime} = \frac{1}{8}(9 - 3/2\sqrt{t})$$

$$y^{*\prime} = \frac{3}{8}(3/2\sqrt{t} - 1)$$

These change sign at $t = 1$ and $t = 4$. This result is not surprising. Parts a) and b) ask about what happens to x or y when the other one is fixed. You would only expect them both to be increasing with t if π were supermodular. It's not.

4. Note that the integral parts of π are supermodular in $x, t, -a$. Use the **MV-MCST** to see that $x, t, -a$ are non-decreasing in P_a and non-decreasing in $-P_t$. This should mean that the covariation will be positive. I'll discuss this in class, if there's enough time.

5. None of these are cost functions. (a) fails because it isn't homogeneous. (b) and (c) fail because they are decreasing in w. (d) is not concave.

6. You'll find that:

$$l^{1-\alpha-\beta} = p\left(\frac{\alpha}{r}\right)^{\alpha}\left(\frac{\beta}{w}\right)^{1-\alpha}$$

and something rather similar for capital.
If $\alpha + \beta > 1$, then the firm either wants to close down, or it wants to produce an unlimited amount. If $\alpha + \beta = 1$, then there are constant returns to scale, and the firm wants to produce zero or infinity, or it doesn't care how much it produces.