Problem Set III

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1. Prove that, if u is a von Neumann-Morgenstern utility function representing preferences $\mathcal{R}$ over $\mathcal{P}(X)$ then $v(x) = au(x) + b$ is also a vN-M utility function representing the same preferences for all $a > 0, b \in R$.

2. Suppose a consumer has vN-M utility u over terminal wealth, initial wealth W, and faces a loss L with probability π. The consumer may buy as much or as little insurance as he or she likes at cost P per dollar insured (i.e. to fully insure against a loss R, the premium is PR)
   (a) Which lotteries is the consumer choosing between? Remember that a lottery is a vector of *outcomes* and the *probabilities* of those outcomes.
   (b) Formulate the consumer's utility maximisation problem for deciding how much insurance to buy, and find the first order conditions.
   (c) What is the expected profit of the insurance company (assuming that losses are the only costs)?
   (d) If the insurance market is competitive (zero profit), show that insurance is actuarially fair. Why might you not expect insurance to be actuarially fair in real markets, even if they are competitive (aside from cost of capital and operating expenses)?
   (e) With actuarially fair insurance, show that the consumer will always insure fully if he or she is risk-averse.

3. For a vN-M utility u over wealth, we define the Arrow-Pratt Absolute Risk Aversion, A to be:
$$A(w) = -\frac{u''(w)}{u'(w)}.$$
   (a) Show that the family of utility functions $-e^{-rw}$ has constant absolute risk aversion (CARA). Interpret this.
   (b) If $\grave{X}$ is a normal random variable $X\tilde{\ }N(\mu, \sigma^2)$, then $\mathsf{E}[e^X] = e^{\mu+\sigma^2/2}$ (this is just the MGF, by the way). If a consumer has CARA vN-M utility $u(w) = -e^{-rw}$, what is $\mathsf{E}[u(w_0 + X)]$?
   (c) Suppose such a CARA consumer chooses between two lotteries $w_0 + X_i$, where $X_i\tilde{\ }N(\mu_i, \sigma_i^2)$. Write down the conditions under which each is preferred. Suppose the consumer wins the Nobel Prize, so his or her new wealth is $w' = w_0 + \$1.2\text{m}$. Will his or her choices change?

4. Imagine that you are choosing between two lotteries:
(a) \$24,000 for sure
(b) .33 probability of \$27,500, .66 chance of \$24,000, .01 chance of nothing.
Which would you choose? In experiments (some with real money), most people would take the sure bet. Now consider the choice between another two lotteries:
(c) .33 chance of \$27,500, .67 chance of nothing.
(d) .34 chance of \$24,000, .66 chance of nothing.
Which would you choose? Most people choose the first lottery. Prove that this violates the vN-M axioms. This is the *Allais Paradox*. Interpret.

5. An urn contains 300 marbles; 100 red, and 200 some mixture of blue and green. You pick one at random.
(a) You get \$1000 if the marble you pick is of a specified colour. Would you specify blue or red?
(b) You get \$1000 if the marble selected is *not* of a specified colour. Would you specify blue or red?
Most people pick red in the first lottery, and red in the second as well, with strict preference. Show that this behaviour is paradoxical, if the decision maker assesses probabilities in a reasonable way. This is called the *Ellsberg Paradox*. Interpret.