14.121 Fall 1999, Problem Set 1
due Thursday, September 23

1. Prove each of the following statements.

   (a) $(x\mathbf{I}y) \Rightarrow (y\mathbf{I}x)$

   (b) $\forall x,\ (x\mathbf{I}x)$

   (c) $(z\mathbf{I}y) \wedge (y\mathbf{I}z) \Rightarrow (z\mathbf{I}x)$
   These conditions say that $\mathbf{I}$ is an *equivalence relation.*

   (d) $(x\mathbf{R}y) \wedge (y\mathbf{P}z) \Rightarrow (x\mathbf{P}z)$

   (e) If u represents the preferences $\mathbf{R}$, then $\mathbf{R}$ is (weakly) monotonic if and only if u is nondecreasing. (You have to prove *both* directions.)

   (f) If u represents $\mathbf{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $\phi \circ u$ represents $\mathbf{R}$.

2. Assume $\mathbf{R}$ satisfies axioms A1-A3 and let X be a finite set. Show that the elements of X can be ordered so that the order corresponds to the ranking according to $\mathbf{R}$, that is, we can write them

$$\ldots x^2 \mathbf{R} x^1.$$

   Hint: use induction on N, the number of elements in X.

   Extra credit: Show that a similar procedure gives a utility function for $\mathbf{R}$ on any countable set X. Note that the range of the utility function might not be a subset of the integers in this case.

3. Consider a consumer with preferences over two goods. The preferences are lexicographic: $(x_1, x_2)\mathbf{R}(y_1, y_2)$ whenever $x_1 > y_1$; if $x_1 = y_1$, $(x_1, x_2)\mathbf{R}(y_1, y_2)$ if $x_2 \geq y_2$.

   (a) For a given $\mathbf{x}$, sketch the set of bundles weakly preferred to $\mathbf{x}$.

   (b) Are the preferences complete? Reflexive? Transitive? Weakly monotonic? Locally non-satiated? Are the better-than sets closed? Prove your answers.

   (c) Can these preferences be represented by a continuous utility function? (Sketch a proof of your answer).

4.

   (a) Show that the function $u(x) = \lambda_x$ constructed in the proof of the representation theorem in fact represents the preferences **R**.

   (b) Show that if **R** is not complete, reflexive, and transitive, then there is no function $u : X \to \mathbb{R}$ which represents **R**.

5. Show that the Substitution and Archimedean axioms are *necessary* for an expected utility representation.

6. Consider a population of (potential) taxpayers, all of whose preferences over wealth lotteries satisfy the von Neumann-Morgenstern axioms. They all have the same vN-M utility function, $u(\cdot)$, which is strictly increasing and concave. You can also assume u is differentiable as many times as you like.

   Each citizen has wealth w, which is subject to taxation at rate $\tau < 1$. The goverment requires each citizen to report her wealth, but the citizens can lie. For simplicity, assume they must either report their true wealth or zero. Since paying no taxes is a good deal, the government must audit some of the people who report zero. Assume that wealth is continuously distributed and that everyone has some wealth.

   The government implements the following auditing scheme: anyone who reports zero is audited with probability p. If the audit determines the citizen did in fact have positive income (it always will), she forefits all of it to the tax collector.

   Assume that the (utility) return to tax evasion is increasing in wealth. Write a simple expression which determines which citizens report their true income and which report zero. Use this expression to deterimne how the set of people who cheat varies with τ and p.