14.121 Fall 1999, Problem Set 3 Professor Susan Athey, T.A. Scott Ashworth

Due Friday, October 8, 1999

1. Consider two famous applications of the envelope theorem in consumer theory, Shephard's Lemma and Roy's Identity. Formally state and prove these results. Then, provide a verbal interpretation of each.

2. A Giffen good is a good for which Marshallian demand goes up when its price goes up.

   (a) Draw a diagram illustrating this possibility, and relate the diagram (very precisely) to the Slutsky equation.

   (b) Formally prove the following: (i) For a good to be Giffen, some normal good must be displaced by the inferior good as the price increase lowers real income. (ii) A Giffen good must also be an inferior good (that is, a good whose demand falls with income).

   (c) Many intermediate microeconomics texts mention the Irish potato famine as an example of a Giffen good. Specifically, they argue that the price of potatoes went up, but people consumed more potatoes, during this famine. Consider the following model of the famine. Graph the budget constraint facing a person with income Y in a two-good economy, potatoes and bread, where the price of bread is one and the price of potatoes is $\$P$. Suppose that potatoes are a Giffen good in the relevant range, that is, the demand for potatoes rises with price when income is held fixed. Show how the budget constraint and optimal bundle change when the price rises from P^1 to P^2, for each of the following assumptions. (i) Y is independent of the price; (ii) $Y = S \cdot P$, where S is fixed; (iii) $Y = S \cdot P$, where we first consider $P = P^1$ and $S = S^1$, and then we consider $P = P^2$ and $S = S^2$, where $S^2 < S^1$ (price rises and supply falls). Explain how the trade-off between potatoes and bread changes in (i)-(iii). Given that the production of potatoes fell drastically during the Irish potato famine, what does this say about the Irish potato famine as an example of the kind of experiment which might be used to determine whether a good is Giffen?

   (d) In what types of real-world settings might you expect to look for a Giffen good? How could you avoid some of the pitfalls implicit in part (c)?

3. A consumer in a three good world faces initial prices $p_1 = p_2 = p_3 = 1$. She buys $x_1 = x_2 = x_3 = 2$. Prices change, and in the following year, thee consumer faces prices $p_1 = p_3 = 4$, and $p_2 = 2$. She buys $x_1 = 1$, $x_2 = 2$, $x_3 = 10$.

   (a) Construct Paasche and Laspeyres price indicies.

   (b) What can you say about the statement that "The Layspeyers index bounds the true cost of living from above, while the Paasche index bounds it from below"? Prove your answer algebraically, and illustrate it using a carefully labelled diagram (a 2-good diagram is fine).

   (c) Given the possible answers to (b.), what can you say about whether the consumer is better or worse off in the second period than the first?

4. A sales agent travels from city to city in the course of her work. To compute the size of her expense account cap, her firm uses the following procedure: The cap in the first ("home") city is fixed at x dollars. Then in each subsequent city, the cap is set equal to the amount of money just equal to the amount needed to buy the old city's bundle at the new city's prices. Assume that the cap is binding and that relative prices vary from city to city.

   Does the agent's utility increase, decrease, or vary non-monotonically as she moves from city to city? Relate your answer to what you know about the Laspeyres price index.

5. During the recent debate over revising the CPI, Robert Kuttner offered several arguments against adjustment in a Business Week column. Below are two quotes. In each case, explain why his logic is misguided. Use a diagram or algebra to back up your point.

   (a) "[S]ince the CPI uses a fixed 'basket' of goods, it misses what economists call substitution effects. If Steak goes to \$5 a pound, thrifty consumers switch to chicken. Critics say the CPI thus overstates what consumers actually spend. Yes, but then chicken isn't steak. If it's fair to adjust for quality gains, let's also count quality losses."

   (b) "[Critics say that...] the BLS waits years before including new products in the standard basket, thus missing some dramatic price reductions. The pocket calculator that first cost \$1000 now sells for \$10. True, but...few people bought \$1000 calculators."

6. This problem is designed to guide you through some calculations related to the deadweight loss of commodity taxation and the determination of the optimal tax rate.

   Consider a consumer who buys n goods, with taxes imposed on the goods. The pre-tax price vector is $\mathbf{p}$, the consumer's pre-tax income is Y and realized utility is $\bar{u}$, and the post tax prices are $\mathbf{q}$. We define the compensated tax revenue as $T(\mathbf{q}, \mathbf{p}, \bar{u}) = (\mathbf{q} - \mathbf{p}) \cdot \mathbf{h}(\mathbf{q}, \bar{u})$.

   (a) Explain why $T(\mathbf{q}, \mathbf{p}, \bar{u})$ gives the revenue that the government would collect if the consumer were given a lump-sum transfer which put her back on the pretax indifference curve.

   (b) In the special case of a two good economy, draw a diagram showing the indifference curve associated with u and the pre and post tax budget lines, assuming perfect compensation. Normalizing the price of good two at 1 and assuming only good one is taxed, illustrate the iso-expenditure lines (budget lines) corresponding to $E(\mathbf{p}, \bar{u})$ and $E(\mathbf{q}, \bar{u})$. In particular, label the expenditures corresponding to $x_1 = 0$ (on the x_2 axis). Then show the compensated tax revenue, $T(\mathbf{q}, \mathbf{p}, \bar{u})$, on the x_2 axis. (Hint: Draw another iso-expenditure line through $\mathbf{h}(\mathbf{q}, \bar{u})$, but at prices $\mathbf{p}$. This is done in most intermediate texts, if you get stuck.)

   (c) Define the deadweight loss of the tax by $L(\mathbf{q}, \mathbf{p}, \bar{u}) = E(\mathbf{q}, \bar{u}) - Y - T(\mathbf{q}, \mathbf{p}, \bar{u})$. Illustrate this quantity on the x_2 axis of your diagram. Why is this an appropriate definition of deadweight loss? Be precise.

   (d) For the rest of the question, return to the general case of n goods. Use the definitions of Y, $E(\mathbf{q}, \bar{u})$, and $T(\mathbf{q}, \mathbf{p}, \bar{u})$ to show that $L(\mathbf{q}, \mathbf{p}, \bar{u}) \geq 0$. What is the interpretation of this? (Don't just say there is deadweight loss. One of the advantages of a formal definition is that you can say exactly what that means.)

(e) Write out $\frac{\partial}{\partial q_i}E(\mathbf{q}, \bar{u})$, $\frac{\partial}{\partial q_i}T(\mathbf{q}, \mathbf{p}, \bar{u})$, and $\frac{\partial}{\partial q_i}L(\mathbf{q}, \mathbf{p}, \bar{u})$ as functions of $\mathbf{t} = (\mathbf{q} - \mathbf{p})$ and S_{ij}, the elements of the Slutsky matrix for $E(\mathbf{q}, \bar{u})$.

(f) Now use your previous work to study optimal commodity taxation. Assume the government must raise revenue equal to R. It decides to do this in the way that minimizes dead-weight loss, so it solves

$$\min_{\mathbf{q}} L(\mathbf{q}, \mathbf{p}, \bar{u}) \text{ subject to } T(\mathbf{q}, \mathbf{p}, \bar{u}) = R.$$

Use the Kuhn-Tucker conditions and your answers above to show that the solution must satisfy, for all k,

$$-\frac{\sum t_j S_{jk}}{h^k(\mathbf{q}, \bar{u}) + \sum t_j S_{jk}} = \lambda.$$

Interpret this expression, and give an intuition about why it describes the optimal taxation policy. (Hint: go back to the intuition from the solution to the standard consumer utility maximization problem, and think about how we interpreted the optimality conditions there. Try to mimic the language, but of course here the objective and constraint are different!)

(g) Suppose that, out of the set of $n > 3$ possible goods, only goods 1-3 are taxed. Further, suppose that $S_{12} = 0$, $S_{23} = 0$, and $S_{13} = 0$. Interpret this assumption. (Why do we need $n > 3$ to make it legitimate?) Then, rewrite the formulae for the optimal taxes on goods 1-3 in terms of the percentage tax (t_k/p_k) and the own-price elasticities of h^k. Carefully and precisely interpret your answer.

(h) What do your answers in the last two parts tell you about how the tax code should be designed? Give examples in terms of real-world commodities.