Problem Set 4 Professor Susan Athey

Due Monday, October 18

Instructions. These problems are intended to give you practice on the comparative statics theory and applications we discussed in class. The more theoretical questions are for you to learn from; when studying for the exam, you should focus on the economic applications, which are important.

For the economic applications, just apply the comparative statics theorems, you don't have to rederive anything. The point is that the theorems make all of the economic questions extremely easy! For examples of that principle, see especially #3, 4, and 5 below. In each, however, it is useful to pause and think about the results, what they mean, and why they are true. #6 and #7 are more substantive.

When I ask comparative statics questions without specifically making reference to the set of optimizers, you may simply give results which apply to the highest element of the set.

1. This question helps you understand the increasing differences property. Suppose that $h : \Re \to \Re$.
   (a) Let g be nondecreasing. Show that if $f : \Re^2 \to \Re$ satisfies increasing differences, so does the composite function h defined by $h(x, y) = f(g(x), y)$. Interpret this in words. Intuitively, why is this a desirable property when we are thinking about comparative statics?
   (b) Let h be twice differentiable. Prove that $f(x, \theta) = h(x - \theta)$ has increasing differences if and only if h is concave.

2. Strict monotonicity can be important for some economic analyses, in particular in information economics and contract theory. Suppose that f satisfies strictly increasing differences. That is, suppose that for all $x'' > x'$, $f(x'', \theta) - f(x', \theta)$ is strictly increasing in θ. Let $X^*(\theta) = \arg\max_{x \in \Re} f(x, \theta) + g(x)$ be nonempty for each θ.
   (a) Show that for $\theta'' > \theta'$, if $z \in X^*(\theta')$ and $y \in X^*(\theta'')$, then $y \geq z$. [Hint: start by writing down the inequalities which must be satisfied in order for the two choices to be optimal (and particular, be preferred to the other choice under consideration) at the respective parameter values. Try to combine the inequalities.]
   (b) Now suppose that f is differentiable in x, so that $\frac{\partial}{\partial x} f(x, \theta)$ is strictly increasing in θ, and suppose g is differentiable too. Show that in part (a), $y > z$. [Hint: use the first order conditions.] Extra: can you draw a picture of $f(x, \theta')$ and $f(x, \theta'')$ as functions of x that show how this result fails if f is not differentiable in x at z (for example, the function is kinked there)?

3. This question applies the results of the last one. Suppose that a worker's cost of going to school is $c(x, \theta)$, where x is the amount of school and θ is the worker's ability/productivity.
   (a) Suppose that c is differentiable and $\frac{\partial^2}{\partial x \partial \theta} c(x, \theta) \leq 0$. Interpret this condition in words in the context of the model.

(b) Now suppose that firms can observe education but not ability, and thus offer wages $w(x)$ which depend only on education. The worker's utility is given by $w(x) - c(x, \theta)$. Is there any wage function which will induce higher ability workers to choose lower levels of education?

(c) Now suppose that $\frac{\partial^2}{\partial x \partial \theta} c(x, \theta) < 0$. Based on your answers above, what is a sufficient condition on $w(x)$ such that two workers of different abilities choose different levels of education?

(d) Conclude from this that even if education is unproductive, firms may be willing to pay higher wages for higher levels of education.

4. For this question, you need to use the fact that maximizing $\pi(x, \theta)$ is the same as maximizing $\ln(\pi(x, \theta))$. Let $Q(p, \theta)$ be a downward-sloping demand curve, let θ be a parameter of the demand curve, and suppose that Q is differentiable in p and strictly positive in the relevant region of prices.

   (a) Show that demand becomes more inelastic with θ if and only if $\ln(Q(p, \theta))$ has increasing differences.

   (b) Suppose that the firm solves $\max_p (p - c) Q(p, \theta)$. Derive a sufficient condition for p to increase (weakly) with θ and relate your answer to (a).

   (c) How does price change with cost c? Does this depend on any assumptions about the demand curve?

   (d) It is often assumed that the demand curve is concave in p, in order to validate the approach based on first order conditions. Show that concave demand is sufficient to guarantee that the profit function is concave. Can you give an economic justification for concave demand? Is this necessary for the comparative statics prediction?

5. Suppose that there are two inputs to production, technology and the skill level of a single worker who uses the technology. There are two technologies available for production, one which uses computers and the other which uses old-fashioned technology. If the firm chooses the computer-driven technology, its production function is $G(s)$, while the production function is $H(s)$ if it chooses the old technology. The total cost of skill s is $c(s)$ on the market.

   (a) What are sufficient conditions on G and H to guarantee that profit-maximizing firms who use the computer-driven technology buy more skill than firms who use the old-fashioned technology? If we wish the result to hold without any structure on $c(s)$, is there a weaker condition which will do? [Hint: Define a function $f(s, \theta)$, where $f(s, \theta^H) = G(s)$ and $f(s, \theta^L) = H(s)$.]

   (b) Suppose that the cost of the computer-driven technology falls smoothly over time, starting at a very high level and eventually reaching zero, and that firms choose their level of technology and skill at the same time. Assume whatever conditions you derived in part (a). Do we expect the skill level selected by the firms to change smoothly? [Your answer here may be informal; we will work on formal results about two inputs next week.]

6. Let $F(k, l; \theta)$ be a production function, where θ is a parameter describing the production function. Suppose that the firm purchases inputs on perfectly competitive input markets and minimizes cost subject to an output constraint, q.

(a) Suppose that the production function is differentiable. How must the parameter change the production function in order to guarantee that the cost-minimizing choice of capital increases with θ? Provide an interpretation. (Hint: define the isoquant function $L(k; q, \theta)$ and use the implicit function theorem to determine its properties.)

(b) How does your answer change if the isoquants are not convex, or if capital must be purchased in discrete units?

7. Suppose that a firm uses labor as an input, but that the firm has market power in the labor market. Thus, the wage rate depends on the quantity of labor used according to $W(l)$, which is strictly increasing. The firm maximizes profits: $\max_l F(l) - W(l) \cdot l$, where F is increasing. You may assume it is concave as well if you like, though that is not necessary for the analysis. [NOTE: This problem has some interesting economics. Think about it, don't just plod through mechanically!]

   (a) Plot the "labor demand curve," $F_l(l)$, as a function of l. Then plot the "labor supply curve," which shows wages as a function of the quantity of labor purchased. On the same picture, plot the "marginal labor cost" curve, which shows $\frac{d}{dl}[W(l) \cdot l]$. How do the two curves compare at any choice of l? Interpret this. [Note: if you remember analyzing monopoly problems, this is like a marginal revenue curve.]

   (b) Use the first-order conditions to show that the optimal choice of labor equates "labor demand" and "marginal labor cost," and illustrate that on your diagram. How does the quantity of labor demanded compare to the "competitive" outcome, where $F_l(l^C) = W(l^C)$? [Note: this is again like a monopoly problem.]

   (c) Suppose that the government imposes a minimum wage, $w^{\min}$. Show how this changes the "labor supply curve" and the "marginal labor cost" curve. Be careful here! Can the new "marginal labor cost" curve cross the old one?

   (d) Use the increasing differences comparative statics theorem to evaluate the statement "when a minimum wage is imposed, firms always use less labor, at least weakly." To do this, define an auxiliary function, $h(l; \theta)$. Let $h(l; \theta') = \min(W(l), w^{\min}) \cdot l$, and let $h(l; \theta'') = W(l) \cdot l$. Using your answer to (c) to help you, can you say whether h has increasing differences? Use a graph to illustrate your result about the possible effects of the minimum wage on the optimal choice of labor.