Problem Set 5 Professor Susan Athey, TA Scott Ashworth

Not to be graded

Instructions. This last set of problems is intended to give you practice for the exam. You do not need to hand in these problems. For the purposes of this problem set, let me formalize a theorem we discussed briefly in class. Define:

$$\mathbf{x}^*(\theta) = \arg\max_{\mathbf{x}\in\Re^n} f(\mathbf{x},\theta).$$

Suppose for simplicity that $\mathbf{x}^*(\theta)$ is nonempty. Then if f is supermodular (that is, it satisfies increasing differences in each pair of variables, or, if f is differentiable, this requires $\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x},\theta) \geq 0$ for all $i \neq j$, and $\frac{\partial^2}{\partial x_i \partial \theta} f(\mathbf{x},\theta) \geq 0$ for all i), there exists a highest (component by component) optimizer, $\mathbf{x}^H(\theta)$. Further, for $\theta'' > \theta'$, $x_i^H(\theta'') \geq x_i^H(\theta')$ for all i. That is, when θ increases, all of the choices go up together.

1. Consider the following profit function (which is differentiable as well as concave in (x,y)):

$$\pi(x,y) = tx + \sqrt{t}y - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{3}xy$$

   (a) Suppose that y is fixed. The firm chooses x to maximize profits. Is the optimal choice of x increasing or decreasing in t?

   (b) Suppose that x is fixed. The firm chooses y to maximize profits. Is the optimal choice of y increasing or decreasing in t?

   (c) Now, suppose that the firm chooses both x and y. Solve for the optimal choices of x and y as a function of t when both x and y are choice variables. Calculate the derivatives of x^* and y^* with respect to t. At $t = 1$, do the optimal choices of x and y increase or decrease as a result of a small increase in t? What about at $t = 4$?

   (d) Interpret your answers in the context of the monotone comparative statics theorems.

2. (From the Fall 1996 final exam). Consider a monopolist who has three choice variables: q, the quantity to sell; "marketing," m, and new equipment, e. The cost of producing q units of output is given by $c \cdot q/e$, where c is a constant. Other input costs: Equipment is purchased on a competitive market at price r. The cost of m units of marketing is given by $h(m;\theta)$, where θ is a parameter which increases the incremental cost of marketing (such as advertising fees). That is, h satisfies increasing differences in $(m;\theta)$. Demand is described by $P(q;m)$, so that total revenue is given by $q \cdot P(q;m)$. Assume that $P(q;m)$ is differentiable as many times and in as many ways as you would like.

   (a) Write down conditions on the function P which guarantee that marketing increases marginal revenue.

   (b) Assume that your conditions from (a) hold throughout the rest of the problem. Note that the firm's profit function is given as follows:

$$q \cdot P(q;m) - c \cdot q/e - r \cdot e - h(m;\theta)$$

Suspose that in the short run, equipment is fixed. How do the firm's short run choices of quantity and marketing change with the parameter θ? With the fixed level of equipment?

(c) How does the firm's long run choice of equipment change with θ?

(d) In response to a 10% increase in θ, will the choices of quantity and marketing change by more in the short run or in the long run? Describe the intuition. For full credit, sketch a proof of your answer.

(e) Derive an expression indicating how the firm's long run profits change with r. Does your answer change with the parameter θ? If so, in what direction? Interpret.

3. In the classical theory of the firm, where firms have a production function F, sell output at price p, and purchase inputs to maximize profits, prove rigorously that profit maximization implies that inputs are chosen so that input costs are minimized for the level of output chosen by the firm.

4. A certain market has a demand curve given by

$$Q = 1100 - 50P.$$

There are 100 firms, each with cost function

$$C(q) = \frac{1}{2}q^2 + 10q + 5.$$

(a) Find the (short-run) market equilibrium for this market, and display it graphically with demand and supply curves.

(b) Now suppose the government imposes a \$3 per unit tax. How does this change the market equilibrium?

(c) How is the burden of the tax shared between consumers and producers? What is the excess burden (also known as deadweight loss)? Answer both algebraically and geometrically.

(d) What if instead of 100 firms, there was only one, with the same cost function. What is the monopoly level of output? What is the deadweight loss? Does the monopolist have a supply curve? Explain your answer.

5. In the classical theory of the firm, where capital is fixed in the short run, prove formally that the short-run average cost function is tangent to the long-run average cost function. Relate your answer to the envelope theorem.