Choice Under Uncertainty

Let X be a finite set of bundles (simplifies analysis). Let N be the number of bundles.

Observe: to get utility representation over finite bundle we need only Axioms 1-4, which would allow us to rank the whole set.

We could have $x \in \mathbb{R}^k$ (vectors)

$(2, 1)$

$(1, 2)$

or could have $x \in \mathbb{R}$, \$

Typically in utility theory we work with utility in dollars but there is nothing special about that.

Lottery in discrete world: $L(X)$ is the set of vectors $p \in [0,1]^N$, where $p_i \in [0,1]$, $i = 1, ..., N$, and $\sum_{i=1}^{N} p_i = 1$.

Example: Let the set of bundles be $X = \{x_1,\ x_2,\ x_3\}$.

The lottery $p = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$ can be interpreted as follows:

- Bundle x_1 with probability $\frac{1}{4}$, and
- Bundle x_2 with probability $\frac{1}{2}$, and
- Bundle x_3 with probability $\frac{1}{4}$.

Our goal: use a utility function defined on goods to represent utility over lotteries.

- We still want to be able to represent idea that we get x_i with probability 1.

Notation:

$\delta_i = (...0, 1, 0, ...)$ i th component, so for every good, there's a δ_i saying I can get x_i for sure. So all "for-sure" bundles can be represented in this way.

- Old: axioms on **R** over goods $\Rightarrow \exists u : X \rightarrow \mathbb{R}$
- New: Axioms on **R** over lotteries $\Rightarrow \exists\ v :\ L(X) \rightarrow \mathbb{R}$

Suppose **R** is a relation over lotteries. Example: $p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $q = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ (Observe: all vectors add to one, so monotonicity is not a consideration in this context).

We know if we consider a finite set of lotteries, then if **R** is complete, transitive, reflexive $\Rightarrow \exists\ v\ s.t.\ v(p) \geq v(q) \Longleftrightarrow\ p\ \mathbf{R}\ q$.

How do we relate these preferences over bundles? Relate this preference relation v to preferences over goods, i.e., $u(x_1),\ u(x_2),\ u(x_3), ...$

<u>Goal:</u> Show $\exists\ u : X \rightarrow \mathbb{R}\ s.t.\ v(p) = \sum_{i=1}^{n} p_i \cdot u(x_i)$

Find Axioms over preferences over goods such that v has an expected utility representation. v is linear in the probabilities, but <u>not</u> in goods. That is, it's not same as saying the agent is risk neutral.

Example: $X = \{\$10, \$20, \$30\}$,

Find u s.t. $v((.3, .4, .3)) = .3u(10) + .4u(20) + .3u(30)$.

No restriction yet, could be linear or not.

Let $p, q \in \; L(X)$, $\alpha \in [0,1]$ $r = \alpha\, p + (1-\alpha)\, q$, α is scaling factor; p, q vectors.

$\alpha = .5, \qquad p = (.3, .4, .3), \qquad r = (.2, .65, .15), \qquad q = (.1, .9, 0)$

What are properties of $\mathbf{R}$ (preference relation over lotteries) *s.t.* there exists an **expected utility representation**, that is,

$$\exists\, u : X \to \mathbb{R}, \;\; s.t.\; p\mathbf{R}q \Longleftrightarrow \sum p_i\, u(x_i) \geq \sum q_i\, u(x_i).$$

Back to Axioms...

First Complete (A1), Reflexive (A2), and Transitive (A3).

so

- $p\mathbf{R}q$ or $q\mathbf{R}p$
- $p\mathbf{R}p$
- $p\mathbf{R}q$ and $qRr \Rightarrow p\mathbf{R}r$

Can say this about lotteries just as one could say this about goods.

New Axioms

E1 Substitution (independence) Axiom

Suppose $p\mathbf{R}q$. Then, for any $\alpha \in [0,1]$ and any $r \in \ L(X)$,

$$\alpha\, p + (1-\alpha)\, r \ \ \mathbf{R}\ \alpha\, q + (1-\alpha)\, r.$$

example:

$p = (.5, .5, 0) \qquad q = (0, .5, .5)$

$\alpha = \frac{1}{2} \qquad r = (.3, .3, .4)$

$p\mathbf{R}q$ implies:

$$(.4, .4, .2)\ \ \mathbf{R}\ \ (.15, .4, .45)$$

This is a very strong Axiom!

E2 Archimedian Axiom

Consider $p, q, r\ \ s.t.\ p\mathbf{P}q\mathbf{P}r$ (need strong preferences!)

Then, $\exists\ \alpha, \beta \in\ (0,1)\ \ s.t.$

$$\alpha\, p + (1-\alpha)\, r\ \mathbf{P}\ q\ \mathbf{P}\ \beta\, p + (1-\beta)\, r$$

Typically high α and low β. No matter how good p is and how bad r is, you can weigh them so that this holds.

Let $p = \$10,000$ for sure. I'll take the small chance of r to get p.

$q = \$10.00$ for sure.

$r =$ Hit by car for sure.

Definition $u : X \to \mathbb{R}$ is a vonNeumann-Morgenstern expected utility representation $(vN - M)$ of preference relation R (over lotteries $L(X)$) if the following holds:

$$p\mathbf{R}q \;\Leftrightarrow\; \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} q_i u(x_i)$$

VN-M Representation Theorem Let X be a finite set. Given a relation $\mathbf{R}$ on $L(X)$ if A1-A3 and E1 and E2, $\exists$ an expected utility representation of R, u: $X \to \mathbb{R}$.

i.e., if $v: \; L(X) \to \mathbb{R}$ represents $\mathbf{R}$, there exists a u such that $v(p) = \sum p_i u(x_i)$

Observe that utility function u is defined over bundles not over lotteries, but it can be used to define utility over lotteries.

Overview of proof approach:

Show that we can find bundles $b, w \in X$ such that, $\forall p \in L(X)$,

$$\delta_b \; \mathbf{R} \; p \; \mathbf{R} \; \delta_w.$$

i.e., $b = 100$, $w = 0$, $\delta_b =$"100 for sure," $\delta_w =$"0 for sure."

Show that for every lottery p, we can find an α such that :

$$\alpha\delta_b + (1-\alpha)\,\delta_w \; \mathbf{I} \; p$$

Proof of Representation Theorem:

Step 1: There exist two bundles, $b, w \in X$, such that $\delta_b \; \mathbf{R} \; \delta_x \mathbf{R} \; \delta_w \;\; \forall x \in X$.

- - * Recall that if preferences over bundles satisfy axioms 1-3, we can order a finite set of bundles in order of weak preference.
    * A similar argument can be used to rank the finite set of degenerate lotteries, $\delta_1, \delta_2, .., \delta_N$ in order of preference. Then, we can find a best and worst degenerate lottery.

Step 2: show that $\forall \alpha, \beta \in [0,1]$,

$$\alpha\delta_b + (1-\alpha)\,\delta_w \;\mathbf{R}\; \beta\delta_b + (1-\beta)\,\delta_w \Leftrightarrow \alpha \geq \beta$$

- - Proof: Use E1 (substitution axiom) several times.
- We know by assumption that $\delta_b \mathbf{R} \delta_w$. Let $\hat{r} = (1-\alpha)\,\delta_w$, $\hat{p} = \delta_b$, and $\hat{q} = \delta_w$. Applying E1 to $(\hat{p}, \hat{q}, \hat{r})$ implies that $\forall \alpha \in [0,1]$,

$$\alpha\delta_b + (1-\alpha)\,\delta_w \;\mathbf{R}\; \alpha\delta_w + (1-\alpha)\,\delta_w$$

- - Now, let $p = \alpha\delta_b + (1-\alpha)\,\delta_w$.
  - Apply substitution axiom again, now define a new $\hat{r}$ so that $\hat{r} = p$, and a new $\hat{p} = p$. E1 applied to $(\hat{p}, \hat{q}, \hat{r})$ implies:

$$\forall (1-\gamma) \in [0,1], \quad \gamma p + (1-\gamma)p \;\mathbf{R}\; \gamma p + (1-\gamma)\,\delta_w \tag{1}$$

  - But substituting,

$$\gamma p + (1-\gamma)\,\delta_w = \gamma\alpha\delta_b + (1-\gamma\alpha)\,\delta_w.$$

- Thus, simplifying (1), we have shown:

$$\forall \alpha, \gamma \in [0,1] \quad \alpha\delta_b + (1-\alpha)\,\delta_w \;\mathbf{R}\; \alpha\gamma\delta_b + (1-\alpha\gamma)\,\delta_w$$

- Since for any $\beta \in [0,\alpha]$, there exists $\gamma \in [0,1]$ such that $\beta = \alpha\gamma$, we conclude that $\forall \alpha, \beta \in [0,1]$ such that $\alpha \geq \beta$,

$$\alpha\delta_b + (1-\alpha)\,\delta_w \mathbf{R} \beta\delta_b + (1-\beta)\,\delta_w.$$

- - Interpretation: If we can rank two degenerate lotteries, we can rank convex combinations of these two lotteries according to the probability weight on the better degenerate lottery, using the substitution axiom. In particular, more weight on the better lottery is preferred. This is how we will construct our index.

Step 3: $\delta_b \mathbf{R}\ p\ \mathbf{R}\ \delta_w \quad \forall p \in L(X)$.

- - Optional exercise: show this is true.
  - Hint: Proceed similar to Step 2, applying E1 repeatedly, keeping $\hat{p}$ fixed at δ_b and mixing in as $\hat{r}$ one degenerate lottery after another until we arrive at p.

Step 4: Show that $\forall p \in L(X)$, $\exists \gamma \in [0,1]$ such that

$$p \mathbf{I} \gamma \delta_b + (1-\gamma)\delta_w. \tag{2}$$

- Proof: We know $\delta_b \mathbf{R}\ p\ \mathbf{R}\ \delta_w \forall p \in L(X)$ by Step 3.
  - if $\delta_b \mathbf{I} p$, let $\gamma = 1$
  - if $\delta_w \mathbf{I} p$, let $\gamma = 0$
  - Take case where $\delta_b\ \mathbf{P}\ p\ \mathbf{P}\ \delta_w$
  - Apply E-2 (Archimedian Axiom), so that $\exists \alpha, \beta \in (0,1)$ such that

    $$\alpha\delta_b + (1-\alpha)\delta_w\ \mathbf{P}\ p\ \mathbf{P}\ \beta\delta_b + (1-\beta)\delta_w$$

  - We know $\alpha > \beta$, so let's try and squeeze them together to arrive at γ.
  - * Let $\lambda_0 = (\alpha+\beta)/2$.
    * If for some λ_n, $p\ \mathbf{I}\ \lambda_n\delta_b + (1-\lambda_n)\delta_w$, we are done.

- * If not, let $\lambda_n = \frac{\alpha_{n-1}+\beta_{n-1}}{2}$.
  * If $\lambda_n \delta_b + (1-\lambda_n)\,\delta_w$ **P** p, let $\alpha_n = \lambda_n$, $\beta_n = \beta_{n-1}$.
  * if p **P** $\lambda_n \delta_b + (1-\lambda_n)\,\delta_w$, let $\beta_n = \lambda_n$, $\alpha_n = \alpha_{n-1}$.
  * Let $\gamma^* = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n$. This limit exists.

Now, consider several cases.

- – First case: if (2) holds for $\gamma = \gamma^*$, we're done.
  – Second case: suppose

$$\gamma^* \delta_b + (1-\gamma^*)\,\delta_w \ \mathbf{P}\ p \quad (\mathbf{P}\delta_w)\,.$$

  – * By Archimedian axiom (E2), $\exists \eta \in (0,1)$ such that

$$\eta\gamma^* \delta_b + (1-\eta\gamma^*)\,\delta_w \ \mathbf{P}\ p.$$

  * Since $\eta\gamma^* < \gamma^*$, we could find some n big enough such that

$$\eta\gamma^* < \beta_n < \gamma^* \Rightarrow p\ \mathbf{P}\ \beta_n \delta_b + (1-\beta_n)\,\delta_w.$$

  – This contradicts monotonicity (Step 2).
- Third case: analogous.

Why is (E2) called the continuity axiom?

- – If α_n drops all the way to γ^* and the bundle is still strictly preferred to p, Archimedian axiom says that if we drop α a tiny bit further, strict preference will still hold.
  – But this contradicts the fact that all β's are worse!

Step 5: for every $x \in X$, find γ_x such that

$$\delta_x \ \mathbf{I} \ \gamma_x \delta_b + (1 - \gamma_x) \, \delta_w.$$

- – This provides an index for each bundle, phrased in terms of the extreme degenerate lotteries, similar to our choice under uncertainty index.
  – Let $u(x) = \gamma_x$.

Step 6: Suppose $p\mathbf{I}q$. Then, for any $\alpha \in [0,1]$ and any $r \in \ L(X)$,

$$\alpha \, p + (1 - \alpha) \, r \ \ \mathbf{I} \ \alpha \, q + (1 - \alpha) \, r.$$

Simply apply E1 (substitution) in both directions.

Step 7: $p \ \mathbf{I} \ \left[\sum p_i u(x_i)\right] \delta_b + \left[- \sum p_i u(x_i)\right] \delta_w$

- – Proof: $p = \sum p_i \delta_i$, since

$$(p_1, ..., p_n) = p_i (1, ..., 0) + p_2 (0, 1, 0, ...) + ...$$

  Further, $\delta_i \ \mathbf{I} \ u(x_i) \, \delta_b + (1 - u(x_i)) \, \delta_w$ by defn.

- Apply Step 6; taking mixtures maintains indifference.

Step 8: $p \ \mathbf{R} \ q \Leftrightarrow \sum p_i u(x_i) \geq \sum q_i u(x_i)$

- Proof: follows from Step 2 and Step 7.