Micro Theory I

Professor Susan Athey E52-252C athey@mit.edu

http://web.mit.edu/athey/www/14.121/

An Axiomatic Development of Utility Functions.

- Overview
  - In most economic models, we start with an agent's utility function. The utility function maps from bundles that the agent might consume, to the real line. The utility function is convenient: it can be maximized and manipulated using standard mathematical tools.
  - The question: Is it valid to reduce to a simple, real-valued function, something as complicated as an agent's preferences over a wide variety of bundles? What does it really mean about the agent's preferences? Are we imposing some hidden or undesirable assumptions when we take this approach?
  - An approach to an answer: analyze the relationship between axioms about an agent's preferences, and the existence of a utility function that "represents" the agent's preferences.
  - The axiomatic approach: specify basic axioms. See what consequences follow.
- The primitives
  - Bundle $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$;
    x_i is quantity of good i. $x_i \in \mathbb{R}$.
    $x \in \mathbb{R}^n$ is vector of quantities.

- Preferences:
  $x\mathbf{R}y$ "x is weakly preferred to y"
  or $x\mathbf{I}y$ "consumer is indifferent between x and y"
  or $x\mathbf{P}y$ "consumer strictly prefers x to y"

Note: "Usual" order of vectors is not "complete" order, ex: $(1,2)$ and $(2,1)$ are not ordered component-wise.

Goal: Map from primitives to utility function.

- Questions: Given axioms on preferences,
  - Can we find the utility function?
  - Is it unique?
  - What properties does it satisfy?

Axioms About Preference Relations

Let the set of available bundles and the consumer's preferences, represented by $\mathbf{R}$, be given.

Define following axioms:

Axiom 1: Complete Order

$$\forall x, y, \ ((x\mathbf{R}y) \vee (y\mathbf{R}x))$$

This implies: $\mathbf{R}$ *cannot* be defined so that $x\mathbf{R}y \Leftrightarrow x_i \geq y_i \; \forall i$.

bundles are only partially ordered in this way.

Axiom 2: Reflexivity

$\forall x, \; (x\mathbf{R}x)$.

Axiom 3: Transitivity

$(x\mathbf{R}y) \wedge (y\mathbf{R}z) \Longrightarrow x\mathbf{R}z$.

Axiom 4: Weak Monotonicity

$x_i \geq y_i$ for all $i \quad \Rightarrow \quad x\mathbf{R}y$ "good are goods"

Axiom 5: Local Non-Satiation

$\forall x$ and all scalars $\delta > 0$, $\exists y$ such that:

1. $\|y - x\| < \delta$ and
2. $y\mathbf{P}x$. "never at a bliss point, even a local one."

Strong Preference!

Exercise (variations on transitivity).Show (a) $(z\mathbf{I}y) \wedge (y\mathbf{I}x) \Longrightarrow z\mathbf{I}x$;

(b) $(z\mathbf{R}y) \wedge (y\mathbf{P}x) \Longrightarrow z\mathbf{P}x$.

Lemma $\forall x, z$, if $z_i > x_i \; \forall i$, then $z\mathbf{P}x$.

Proof:

First, note that $\exists \; \delta > 0$ such that $\forall y$ satisfying $\|y - x\| < \delta, \quad z_i > y_i$.

For some such y, by (A5), $y\mathbf{P}x$.

But, by transitivity (A3) and your exercise, $(z\mathbf{R}y) \wedge (y\mathbf{P}x) \Longrightarrow z\mathbf{P}x$.

Definition For a given x, $S^+(x) = \{y \mid y\mathbf{R}x\}$ "better than bundles."

$S^-(x) = \{y \mid x\mathbf{R}y\}$ "worse than bundles."

Axiom 6 "Continuity of Preferences"

$\forall x$, $S^+(x)$ and $S^-(x)$ are closed.

(i.e. $\forall \{y_n\} \; s.t \; y_n \; \rightarrow \; y$ and $y_n \in S \; \forall n, \; y \in S$).

Consequence: Path from $y \in S^-(x)$ to $z \in S^+(x)$ passes through point w $s.t.$ $w\mathbf{I}x$.

(we will prove this to the extent that we need it below)

Goal: Show exists a utility function $u : X \rightarrow \mathbb{R}$, $s.t.$

$$(u(x) \geq u(y)) \Leftrightarrow (x\mathbf{R}y)\,. \quad (*)$$

Definition Given commodity space X and relation $\mathbf{R}$, $u : X \rightarrow \mathbb{R}$ represents $\mathbf{R}$ if $(*)$ holds.

Note 1: if u represents $\mathbf{R}$, then so does $\phi \circ u$ for any increasing for $\phi : \mathbb{R} \to \mathbb{R}$. (It is unique up to a monotone transformation).

Note 2: Observe that u contains same information as preference relation. But, u maps from vectors into real line, which is totally ordered in the usual way! This is by definition a complete order! If we didn't assume a complete order on **R,** these two representations could not possibly be equivalent.

Similarly, the order $\geq$ on the real line is transitive and reflexive. So we see why these axioms are required for a representation theorem!

Exercise If u represents $\mathbf{R}$, then $\mathbf{R}$ satisfies (weak) monotonicity if and only if u is nondecreasing.

Hint: Write definition of weak monotonicity, write down def. of non-decreasing. Connect the two!

Proposition If X is countable, and Axioms 1-4 are satisfied, $\exists u_{\mathbf{R}}$ which represents $\mathbf{R}$.

Exercise Show Axioms 1-4 imply that $\exists u_{\mathbf{R}}$ which represents $\mathbf{R}$ when $X = \{\mathbf{x}^1, .., \mathbf{x}^M\}$, $\mathbf{x}^i \in \mathbb{R}^n$, $M < \infty$.

Hint: Show that the elements of X can be ranked in order of preference, and then assign utilities according to the rank.

Optional Exercise Show Axioms 1-4 imply that $\exists u_{\mathbf{R}}$ which represents $\mathbf{R}$ when X is countable.

General Problem: Order all possible! In general we don't have a countable X. Commodities are generally indivisible.

Representation Theorem If Axioms 1-6 are satisfied, then **R** can be represented by a continuous function u.

Sketch of Proof

Overview: define $e = (1, ..., 1)$ ("one each").

Show $\forall x, \exists\ \lambda_x\ \ s.t.\ \ \lambda_x e\ \mathbf{I} x$. Let $u(x) = \lambda_x$.

Issues:

- Show for each x, can partition line λe into "better than" & "worse than" sets.
- Sets closed $\Rightarrow \lambda_x$ in both sets.

Proof:

(Notation: $x \geq y \Leftrightarrow x_i \geq y_i\ \forall i; \qquad x > y \Leftrightarrow x_i > y_i\ \forall i$)

Step 1 $\forall \lambda > \lambda', \ \ \lambda e > \lambda' e$.

Lemma implies $\lambda e\ \mathbf{P}\ \lambda' e$. (uses non-sat. (A5), transitivity (A3))

Step 2 Pick $x \geq (0, \ldots, 0)$.

By A4 (weak monotonicity), $x\ \mathbf{R}\ (0, \ldots, 0)$.

Let $m = \max\{x_1, \ldots, x_n\}$.

Choose $\overline{\lambda} > m$.

By Lemma, $\overline{\lambda} e \mathbf{P}\ x$.

Step 3 Show that $\exists\ \lambda_x$ such that $\forall \lambda < \lambda_x$, $x\mathbf{R}\lambda e$, and $\forall \lambda > \lambda_x$, $\lambda e\mathbf{R}x$.

Define $A = \{\lambda : x\mathbf{R}\lambda e\}$, and $B = \{\lambda : \lambda e\mathbf{R}x\}$.

(a) $\forall \lambda' < \lambda''$, we cannot have $\lambda' \in A \cap B$ and $\lambda'' \in A \cap B$ at the same time.

Pf: $\lambda \in A \cap B$ implies $\lambda e\mathbf{I}x$, by definition. But $\lambda' e\mathbf{I}x$ and $\lambda'' e\mathbf{I}x$ implies $\lambda' e\mathbf{I}\lambda'' e$ by your exercise on transitivity. But this violates Step 1.

(b) $\forall \lambda \in [0,1]$, $\lambda \in A$ or $\lambda \in B$.

Pf: Follows by completeness (A1).

(c) $\lambda' \in A$ and $\lambda'' \in B$ implies that $\lambda' \leq \lambda''$.

Pf: Suppose not. Then $x\mathbf{R}\lambda' e\mathbf{P}\lambda'' e\mathbf{R}x$, a contradiction of reflexivity (A2) and transitivity (A3).

(d) Let $\lambda_x = \sup\{\lambda : \lambda \in A\}$.

By parts (a)-(c), $\inf\{\lambda : \lambda \in B\} = \sup\{\lambda : \lambda \in A\}$.

Step 4 Show $\lambda_x e \in S^+(x)$ and $\lambda_x e \in S^-(x)$.

Observe:

$$\lambda e\mathbf{R}\lambda_x e \ \forall \lambda > \lambda_x$$

$$\lambda_x e\mathbf{R}\lambda e \quad \forall \lambda < \lambda_x$$

Take a sequence $\lambda_n \uparrow \lambda_x$.

$$\Rightarrow \lambda_n e \in S^- (x) \; \forall n.$$

By continuity (A6), $\lambda_x e \in S^- (x)$. This implies $x\mathbf{R}\lambda_x e$.

Likewise, if $\lambda_n \downarrow \lambda_x$, then $\lambda_n e \in S^+ (x) \; \forall n \;\; \Rightarrow \;\; \lambda_x e \in S^+ (x)$. This implies $\lambda_x e\mathbf{R}x$.

But $xR \, \lambda_x e$ and $\lambda_x e \, \mathbf{R}x \Rightarrow \lambda_x e \; \mathbf{I} \; x$

Step 5 Let $u (x) = \lambda_x$

Exercise: Show it works! Need transitive, complete, reflexive?

Continuity: harder & tedious.