# Consumer Theory: Pulling the Pieces Together

- Motivation
  - Hicksian demand has nice theoretical properties, can be used to answer welfare questions.
  - Marshallian demand is observable.

- Connecting Marshallian and Hicksian Demand: The Slutsky Equation
  - Approach: recall the identity

$$h^i(\mathbf{p}, \bar{u}) \equiv D^i(\mathbf{p}, E(\mathbf{p},\bar{u}))$$

  - Differentiate the identity $h^i(\mathbf{p}, \bar{u}) \equiv D^i(\mathbf{p}, E(\mathbf{p},\bar{u}))$:

$$\frac{\partial}{\partial p_j} h^i(\mathbf{p}, \bar{u}) = \frac{d}{dp_j} D^i(\mathbf{p}, E(\mathbf{p},\bar{u}))$$

$$= \frac{\partial}{\partial p_j} D^i(\mathbf{p}, Y)\bigg|_{Y=E(\mathbf{p},\bar{u})} + \frac{\partial}{\partial p_j} E(\mathbf{p},\bar{u}) \cdot \frac{\partial}{\partial Y} D^i(\mathbf{p}, Y)\bigg|_{Y=E(\mathbf{p},\bar{u})}$$

$$= \frac{\partial}{\partial p_j} D^i(\mathbf{p}, Y)\bigg|_{Y=E(\mathbf{p},\bar{u})} + h^j(\mathbf{p}, \bar{u}) \cdot \frac{\partial}{\partial Y} D^i(\mathbf{p}, Y)\bigg|_{Y=E(\mathbf{p},\bar{u})}$$

  - Substitute back in for identities to eliminate the $\bar{u}$: $E(\mathbf{p},\bar{u}) = Y$, $\bar{u} = V(\mathbf{p}, Y)$, $h^j(\mathbf{p}, \bar{u}) = D^j(\mathbf{p}, Y)$.

$$\frac{\partial}{\partial p_j} h^i(\mathbf{p}, \bar{u})\bigg|_{\bar{u}=V(\mathbf{p},Y)} = \frac{\partial}{\partial p_j} D^i(\mathbf{p}, Y) + D^j(\mathbf{p}, Y) \cdot \frac{\partial}{\partial Y} D^i(\mathbf{p}, Y)$$

- Recall: since $E(\mathbf{p},\bar{u})$ is concave, $\frac{\partial^2}{\partial p_i^2} h^i(\mathbf{p},\bar{u}) \leq 0$. Thus, the own-price substitution effect is always negative. This can be rephrased in terms of the potentially observable Marshallian demand function as follows:

$$\frac{\partial}{\partial p_i} D^i(\mathbf{p}, Y) + D^i(\mathbf{p}, Y) \cdot \frac{\partial}{\partial Y} D^i(\mathbf{p}, Y) \leq 0.$$

- Integrability
  - Suppose you make up a nice functional form for a demand system.
  - Is it valid?
  - We have established properties that demand functions must satisfy. What properties of demand functions are sufficient to guarantee that demands come from some underlying utility function?
  - If Marshallian demand satisfies $\mathbf{p} \cdot \mathbf{D}(\mathbf{p}, Y) = Y$, $\mathbf{D}(\mathbf{p}, Y) = \mathbf{D}(\alpha\mathbf{p}, \alpha Y)$ for $\alpha \geq 0$, and the corresponding Slutsky substitution matrix, whose $(i, j)th$ element is given by

    $$S_{ij} = \frac{\partial}{\partial p_j} D^i(\mathbf{p}, Y) + D^j(\mathbf{p}, Y) \cdot \frac{\partial}{\partial Y} D^i(\mathbf{p}, Y)$$

    is negative semi-definite and symmetric, it can be "integrated up" to get the indirect utility function, and this can in turn be used to derive the direct utility function.
  - With these conditions, I can check whether my functional forms work without actually performing the integration (which is often quite tricky).
  - Can empirically estimate Marshallian demands, then construct Hicksian demand and do welfare analysis!

- Characterizing Utility-Maximizing Demands Using FOC's: Results from Intermediate Micro

– Suppose that $u$ is continuously differentiable, locally non-satiated, and strictly quasi-concave.

– Then: $\mathbf{D}(\mathbf{p}, Y)$ solves

$$\min_{\boldsymbol{\mu},\lambda} \max_{\mathbf{x}} \quad u(\mathbf{x}) + \lambda[Y - \mathbf{p} \cdot \mathbf{x}] + \boldsymbol{\mu} \cdot \mathbf{x}$$

– If optimum is interior, then the solution is characterized by $\boldsymbol{\mu}^* = \mathbf{0}$, and

$$\nabla u(\mathbf{x}) = \lambda \mathbf{p}.$$

More generally, either

$$\frac{\partial}{\partial x_i} u(\mathbf{x}) = \lambda p_i, \text{ or } x_i = 0.$$

– Interpretations

  * Shadow Value of Income: By the envelope theorem,

$$\frac{\partial}{\partial Y} V(\mathbf{p}, Y) = \lambda.$$

Bang-for-the-buck:

$$\frac{\frac{\partial}{\partial x_i} u(\mathbf{x})}{p_i} = \lambda.$$

* Along an iso-expenditure line, $\mathbf{p} \cdot \mathbf{x}$ is constant. Thus,

$$\mathbf{p}\cdot \left.\boldsymbol{\nabla}\mathbf{x}\right|_{\mathbf{p}\cdot\mathbf{x}=Y} = 0$$

  i.e., the price vector is $\perp$ to the budget line.

  · 2-D example: Along a budget line,

$$p_1 dx_1 + p_2 dx_2 = 0,$$

$$\Longrightarrow \qquad \left.\frac{dx_2}{dx_1}\right|_{\mathbf{p}\cdot\mathbf{x}=Y} = \frac{p_1}{p_2}$$

* Along an iso-utility line, $u(\mathbf{x})$ is constant. Thus,

$$\boldsymbol{\nabla}u(\mathbf{x}) \cdot \left.\boldsymbol{\nabla}\mathbf{x}\right|_{u(\mathbf{x})=\bar{u}} = 0$$

  i.e., gradient of the utility function is $\perp$ to indifference curve.

  · 2-D example: Along an indifference curve,

$$u_1 dx_1 + u_2 dx_2 = 0,$$

  or

$$\left.\frac{dx_2}{dx_1}\right|_{u(\mathbf{x})=\bar{u}} = \frac{u_1(\mathbf{x})}{u_2(\mathbf{x})}$$

* Tangency: $\boldsymbol{\nabla}u(\mathbf{x}) = \lambda\mathbf{p}$.

– Duality

* To see from tangency:

$$\min_{\mathbf{x},\boldsymbol{\mu},\gamma} \quad \mathbf{p}\cdot\mathbf{x} + \gamma[\bar{u} - u(\mathbf{x})] + \boldsymbol{\mu}\cdot\mathbf{x}$$

  FOC for interior optimum is the same: equalize bang-for-the-buck:

$$\mathbf{p} = \gamma\boldsymbol{\nabla}u(\mathbf{x})$$

  Note: $\gamma = 1/\lambda = \frac{\partial}{\partial\bar{u}}E(\mathbf{p},\bar{u})$.