14.129 Fall 1998, Problem Set 4 — Professor Susan Athey, T.A. Svetlana Danilkina

Due Tuesday, November 3, 1998

1. Suppose that I agents want to get tickets for a concert. These I agents are symmetric and risk neutral, and their values for attending the concert are independently, identically distributed on $U[0,1]$ (uniform). There are two tickets available. A mechanism is described by the strategy space for the agents, the allocation rule which determines who gets tickets as a function of the agents' choices, and a corresponding set of transfers.

   Note: If n random variables are distributed uniformly on $[0,B]$, the expected value of the k^{th} highest value is $(n-k+1)\cdot B/(n+1)$.
   Consider the following auction mechanisms:
   (1) Agents submit sealed bids. The bidders who placed the highest two bids receive tickets, and they pay the amount of the third-highest bid. Everyone else pays nothing.
   (2) Agents submit sealed bids. The bidders who placed the two highest bids receive tickets, and they pay their bids. Everyone else pays nothing.

   (a) For each of the two auctions, answer: (i) What is the expected revenue to the auctioneer? (ii) What is the probability that a bidder with value v_i gets a ticket? (iii) What is the expected payment of a bidder with value v_i (unconditionally, i.e., before the auction begins)?

   (b) Now suppose that ticket prices are regulated, so that the ticket office must sell them at a low fixed price (normalized to zero). Further, assume that agents value their time at a monetary rate of $b \cdot t$, where b is the same for all agents and t is time. Two new mechanisms are being considered, described as follows:
   (3) The ticket office annouces that at a specified time, it will award the tickets for free to the first two agents in line. The agents can then choose to line up t hours before the award time. If they are among the first two agents there, they wait until the box office opens and claim the tickets. If there are already two agents in line, then they go home without bearing any costs (that is, it is costless to show up).
   (4) All agents may spend time lobbying the ticket agent to choose them for the tickets. The two agents who spend the most time lobbying receive the tickets. Lobbying is unobservable to the other agents.
   Assume that a symmetric Bayesian Nash Equilibrium exists for each game. For each of (2) and (3), describe the allocation rule for the corresponding direct mechanism.

   (c) Argue briefly that the expected (monetary equivalent) payment made by an agent who realizes type v_i (unconditional on whether or not she receives a ticket) is the same across mechanisms (1)-(4). What does this imply about the expected value of the total expenditure across all I agents in (1)-(4)? What are the essential assumptions and features of the mechanisms which lead to this "revenue equivalence"? Write an equation describing the agent's indirect utility in each mechanism as a function of the probability of winning, and show how this can be used to prove the revenue equivalence result directly.

   (d) In mechanism (3), how long before the award time does an agent with value v_i show up? In mechanism (4), how much time does an agent with value v_i spend lobbying?

2. Consider the following game. There is a buyer and a seller. The buyer's utility is given by $x \cdot v - t^B$, while the seller's utility is $t^S - x \cdot c$. The buyer's value v and the seller's cost c are private information, with prior distributions $F(v)$ on $[\underline{v}, \bar{v}]$ and $G(c)$ on $[\underline{c}, \bar{c}]$. Assume $\bar{c} \geq \bar{v}$, $\underline{c} \geq \underline{v}$, and $\underline{c} < \bar{v}$. Suppose that we consider implementing an efficient allocation, $x(c, v) = 1$ if $v > c$ and $x(c, v) = 0$ otherwise.

   (a) Describe a set of transfers, $t^B(c, v)$ and $t^S(c, v)$ which implement efficiency in dominant strategies, and further, which only specify non-zero transfers if trade takes place. The logic should be closely related to a second-price auction or a "pivotal" mechanism. Show that these transfers essentially give away the "Gains From Trade" twice, and thus are not budget-balanced. Interpret this finding in the context of giving the agents the correct "social" incentives.

   (b) Now consider designing a Bayesian incentive compatible mechanism. Assume that there are no participation constraints. Derive the transfers $\tilde{t}^B(c, v)$ and $\tilde{t}^S(c, v)$ which are BIC and budget-balanced. Explain why budget-balance can be obtained in this case. Then, describe a case where the buyer's expected utility, conditional on her type, is negative.

   (c) Suppose that the agents have participation constraints: the outside option is 0, and each agent must get non-negative expected payoff after observing his own type. Take the expected value of the DIC transfers from part (a), and use them to derive an expression for each agent's expected utility conditional on type. Argue that you have uniquely characterized each agent's expected utility and expected transfer if the participation constraints are binding. Using what you learned from part (a), argue that no BIC, individually rational mechanism implements efficient trade. This is the Myerson-Satterthwaite Theorem.

   (d) Suppose that a principal wishes to profit from a mechanism which implements trade, not necessarily efficient, in this setting. What rule will the principal use to allocate the object? Interpret your answer in the context of "virtual types."

   (e) Suppose that the valuations are both uniformly distributed on $[0, 1]$. Show that the optimal mechanism entails trade if and only if $v - c > 1/2$.

   (f) Continue with the uniform distribution example, and consider the double-auction of Chatterjee and Samuelson (1983): The seller and the buyer simultaneously choose prices p_S and p_B, respectively. If $p_S \leq p_B$, trade occurs at price $p = \frac{1}{2}(p_S + p_B)$. If $p_S > p_B$, there is no trade and no transfer. Show that $p_S(c) = \frac{2}{3}c + \frac{1}{4}$ and $p_B(v) = \frac{2}{3}v + \frac{1}{12}$ is a Bayesian Nash Equilibrium. Calculate the value of the efficiency loss due to private information. Is there a way to reduce this efficiency loss, subject to BIC and individual rationality?

3. Consider the same game as above, but instead assume that v_s and v_b are discrete random variables where

$$v = \begin{cases} a \text{ with probability } p \\ b \text{ with probability } 1 - p \end{cases}$$

$$c = \begin{cases} d \text{ with probability } q \\ e \text{ with probability } 1 - q \end{cases}$$

   Assume that $0 \geq a < d < b < e$.

(a) For which combinations of types is trading efficient?

(b) Consider the following bargaining mechanism in which the seller makes a "take it or leave it" offer, which the buyer must accept or reject. Show that this mechanism does not necessarily lead to inefficient trading.

(c) Relate this result to the Myerson-Satterthwaite Theorem.