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Linear Sharing Rules and Dynamics in Moral Hazard Models

- Example: Mirlees Problem
  - $x = e + \theta, \qquad \theta \sim N\left(0, \sigma^2\right)$
  - There is no solution!
  - To see this, observe:

$$\frac{1}{u'\left(s\left(x\right)\right)} = \lambda + \mu \frac{f_e}{f}\left(x, e\right) = \lambda + \mu \frac{x-e}{\sigma^2}$$

    but if $x = -\infty$, we would require $\frac{1}{u'(s(x))} < 0$.
  - Likelihood ratio unbounded below. Intuition: normal tails are infinitely informative.
  - If $x = -100$, you know you had L.

- This is a technical as well as a substantive problem.
  - Can approximate first best with very bad, unlikely punishments.
  - As punishments get worse, you can ,make it happen less often.
  - Formally, let $u\left(s^*\right) = c\left(e^*\right), \qquad \underline{u} = 0.$

$$s(x) = \begin{cases} s^* \text{ if } x \geq x_0 \\ -k \text{ if } x \leq x_0 \end{cases}, \qquad u(-k) \to -\infty$$

- Idea to Formalize:
  - Mirlees scheme is highly sensitive to knowledge of $F(x \mid e)$, u, etc. "Not robust."
  - Choice set principle is too rich relative to agent.
  - Dynamic setting
    - Quota problem - if did not fulfill quota at end of the month, work inefficiently hard.
    - Solution: linearity may be optimal (constant incentive pressure over time)
- Assumption: no wealth effects CARA

$$\begin{aligned} u(w) &= -e^{-rw}, \quad [nb \leq 0] \\ u(w + \underline{w}) &= -e^{-r(w+\underline{w})} = \left(-\left(-e^{-r\underline{w}}\right)\right)\left(-e^{-rw}\right) = -u(\underline{w})\,u(w) \end{aligned}$$

  - Where you are will not change intensity of incentive pressure or IR constraints.
  - For IR:

$$u(x)\,f(x)\,dx \geq u(0) \Rightarrow$$

$$\int u(\underline{w} + x)\,f(x)\,dx =$$

$$-u(\underline{w}) \int u(x)\,f(x)\,dx \geq -u(\underline{w})\,u(0) = u(\underline{w})$$

– Consequence: if some contract satisfies IR at $\underline{w} = 0$, will also at $\underline{w} > 0$

Exercise: Same argument implies $\underline{w}$ has no effect on IC constants.

Suppose $x \sim N\left(\mu, \sigma^2\right)$, $u = -e^{-rw}$. Let $CE\left(\mu, \sigma^2\right)$ be defined implicitly by

$$u\left(CE\right) = E\left[u\left(\alpha x\right)\right].$$

Exercise: Show $CE\left(\mu, \sigma^2\right) = \alpha\mu - \underbrace{\frac{1}{2} r \alpha^2 \sigma^2}_{\text{risk premium}}$.

Exercise: Suppose: $\mathbf{x} \sim N\left(\boldsymbol{\mu}, \Sigma\right)$, $u\left(CE\right) = E\left[u\left(\boldsymbol{\alpha} \cdot \mathbf{x}\right)\right]$. $\Rightarrow$

$$CE = \sum \alpha_i \mu_i - \frac{1}{2} r \boldsymbol{\alpha}^T \sum \boldsymbol{\alpha}.$$

Aggregation / Linearity

- Setup: Dynamic Model
  - T periods
  - x_1 = outcome in period 1 (sale / no sale)
  - x_T = outcome in period T.

- Assumptions:
  - Agent chooses p_t, probability of success in period t. Can depend on full history.
  - All periods are the same technologically (error structure, cost of effort, etc.)
  - Cost is per-period cost, $c(\cdot)$, and is increasing and convex.
  - Principal: commit to $\underbrace{s\left(x^T\right)}_{\text{full history}}$.
    - Note: we do not necessarily have $s\left((1,0)\right) = s\left((0,1)\right)$.
  - Utility is evaluated at the end of period using CARA preferences (can't quit in the middle)

$$u\left(w\right) = -e^{-rw}$$

    - To interpret the CARA assumption, it applies if the shares are small relative to the total wealth.
  - Take the case where $T = 2$.

- Timeline
  - Period 1: Agent chooses p^1_1, cost $c\left(p^1\right)$ (certainty equivalent unit)

    Agent is choosing the probability of success.

  - Period 2: Agent chooses p^2_i, $\begin{cases} i = 1 \text{ if } x' = 1 \\ i = 0 \text{ if } x' = 0 \end{cases}$ (choice contingent on state of the world)
  - Defn: "Constant incentive pressure" means that the agent is rewarded for success no matter when it occurs.

- Analysis
  - Define function:

$$\omega_i\left(p^2_i, s_{i0}, s_{i1}\right), \quad i = 0, 1$$

    As the solution to:

$$u\left(\omega_i\right) \quad = \quad p^2_i u\left(s_{i1} - c\left(p^2_i\right)\right) + \left(1 - p^2_i\right) u\left(s_{i0} - c\left(p^2_i\right)\right)$$

    This is certainty equivalent from branch associated with outcome i in first period.

  - The incentive constraints:

$$p^1 = \arg\max_{\tilde{p}^1} \quad \tilde{p}^1 \; u\left(w_1 - c\left(\tilde{p}^1\right)\right) + \left(1 - \tilde{p}^1\right) \; u\left(w_0 - c\left(\tilde{p}^1\right)\right) \tag{IC1}$$

$$p^2_i = \arg\max_{\tilde{p}^2_i} \quad \tilde{p}^2_i \; u\left(s_{i1} - c\left(\tilde{p}^1\right) - c\left(\tilde{p}^2_i\right)\right) + \left(1 - \tilde{p}^2_i\right) \; u\left(s_{i0} - c\left(\tilde{p}^1\right) - c\left(\tilde{p}^2_i\right)\right) \tag{IC2i}$$

– Ex-ante IR

$$p^1 p_1^2 \; u\left(s_{11} - c\left(p^1\right) - c\left(p_1^2\right)\right) + p^1\left(1 - p_1^2\right)\; u\left(s_{i0} - c\left(p_1^2\right)\right) \qquad \text{(IR)}$$

$$+ \left(1 - p_1\right) p_0^2 \; u\left(s_{01} - c\left(p^1\right) - c\left(p_0^2\right)\right)$$

$$+ \left(1 - p_1\right)\left(1 - p_0^2\right)\; u\left(s_{00} - c\left(p^1\right) - c\left(p_0^2\right)\right) \geq u\left(0\right)$$

– Principal's problem: Maximize (wrt shares and efforts):

$$p^1\left[p_1^2\left(2x_1 - s_{11}\right) + \left(1 - p_1^2\right)\left(x_1 + x_0 - s_{10}\right)\right]$$

$$+ \left(1 - p^1\right)\left[p_0^2\left(x_0 + x_1 - s_{01}\right) + \left(1 - p_0^2\right)\left(2x_0 - s_{00}\right)\right]$$

subject to: $IR,\; IC1,\; IC21,\; IC20,\; \omega_i\left(p_i^2, s_{i0}, s_{i1}\right) = w_i.$

– Solving:

 * $w_1 - w_0$ determines p^1.
 * IR will bind, determines the intercept of sharing rule; effect of incentives is independent of levels (consequence of CARA)
 * Take desired certainty equivalent levels $w_1 = \hat{w}_1,\; w_0 = \hat{w}_0$ as exogenous (note: like setting IR constraint), find best incentives, p_i^2's in period 2.

$$\text{Max}_{p_i^2, s_{i1}, s_{i0}}\; p_i^2\left(x_i + x_1 - s_{i1}\right) + \left(1 - p_i^2\right)\left(x_i + x_0 - s_{i0}\right) \qquad \text{(AUX)}$$

$$\text{subject to: IC2}i,\; \omega_i\left(p_i^2, s_{i0}, s_{i1}\right) = \hat{w}_i. \qquad (**)$$

Lemma 1 *If $\left(p_i^2, s_{i1}, s_{i0}\right)$ is the solution to the auxiliary program (AUX), p_i^2 and $s_{i1} - s_{i0}$ are independent of $\hat{w}_i$.*

- – FOLLOWS FROM NO WEALTH EFFECTS ASSUMPTION!
    - ∗ Implication of the Lemma: The agent wakes up, he sees a new day, solution does not depend on level of certainty equivalent wealth. $\Rightarrow s_{11} - s_{10} = s_{01} - s_{00}$ in a solution to Principal's problem (same problem in both branches, no matter what w_1 and w_0 are).
    - ∗ Why would they ever be different? For example, if want to implement $p = 0.6$ and wealth effects, may need different incentives in different branches.
  - – Key conclusion: Since incentive intensities and desired efforts all solve the same optimization problem, we have:

Lemma 2 *The principal's 1st period problem is technologically the same as each second period branch.*

PROOF: The expected social surplus following a first-period success can be written in certainty-equivalent units as $\pi(1)$; following a first-period failure it is $\pi(0)$. The only difference for the principal is the desired certainty-equivalent wealth to the agent; but, by Lemma 1, the size of the pie does not depend on this wealth, it just determines the distribution of the pie between principal and agent. Thus, $\pi(1) - \pi(0) = w_1 - w_0$. The principal's first-period problem is:

$$\text{Max}_{p^1, w_1, w_0}\; p_i \left(x_1 + \pi(0) + w_1 - w_0\right) + \left(1 - p_i\right)\left(x_0 + \pi(0)\right)$$

$$\text{subject to} : IC1.$$

Verify that this has the same solution as AUX, up to the choice of w_0.

- – Now, solve the model:
    - ∗ IR will pin down w_0.
    - ∗ For the second-period sharing rules to provide the incentives $w_1 - w_0$ in the first period, need:

$$w_1 - w_0 = \omega_1\left(p_1^2, s_{11}, s_{10}\right) - \omega_0\left(p_0^2, s_{01}, s_{00}\right)$$

    - ∗ The same p is optimal in period 1:

$$p_1^2 = p_0^2,$$

      and best $w_1 - w_0$ satisfies

$$w_1 - w_0 = s_{11} - s_{10} = s_{01} - s_{00}.$$

  – Let a be reward for success. Then:

$$a = s_{11} - s_{10} = s_{01} - s_{00} = w_1 - w_0$$

  (constant incentive intensity). Further,

$$w_1 - w_0 = s_{11} - s_{01}$$

  (consistency, given constant incentives.) Then,

$$s_{11} = s_{10} + s_{01} - s_{00} = 2s_{01} - s_{00}$$

$$\Rightarrow s_{10} = s_{01} = s_{00} + a$$

$$= s_{10} + a = s_{00} + 2a$$

  $\Rightarrow$ fixed reward for success.

* Let $z = \#$ successes. Let x^T be history of successes, success = 1.

  · "Linearity" in outcomes:

$$\underbrace{s\left(x^T\right)}_{\text{whole history}} = s\underbrace{\left(z\left(x^T\right)\right)}_{\#\text{ successes in } x^T} = az + \underbrace{b}_{s_{00}}$$

  · Note: if 3 possible outcomes, scheme will be linear in outcomes, not profits/output.

Aggregation and Stationarity (not really linearity)

- Setup: Brownian Motion

$$dz(t) = \mu dt + \underbrace{\sigma}_{\text{exogenous}} dB(t)$$

  - Note: this is limit of Bernoulli process as $T \to \infty$.
  - Interpretation: $t \in [0,1]$, $z(1)$ is final tally of successes.

The Linear Contracting Model

- Motivate Reduced Form Model
  - Agent: $u(w) = -e^{-rw}$
  - Choose: e
  - Output: $e + \theta, \qquad \theta \sim N(0, \sigma^2)$
  - Contracts are now restricted: $s(x) = ax + b$

- Principal's objective:

$$\int [x - s(x)]\, dF(x \mid e)$$

where $s(x) = ax + b \Leftrightarrow \max_{a,b,e} (1-a)\mu(x \mid e) - b \Leftrightarrow$

$$\max_{a,b,e} (1-a)x - b$$

$$\text{s.t.} \quad \int u(ax + b - c(e))\, dF(x \mid e) \geq u(w) \qquad \text{(IR)}$$

$$e \in \arg\max_{e'} \int u\left(ax + b - c\left(e'\right)\right) dF\left(x \mid e'\right) \qquad \text{(IC)}$$

- Assume $u(w) = -\exp(-rw)$ and $x = e + \theta, \;\; \theta \sim N\left(0, \sigma^2\right)$

  Recall $Eu(z) = u\left(\mu_2 - \frac{1}{2} r \sigma_z^2\right)$, where z is normal, u is exponential

$$\begin{aligned} IR + IC \;\; &\Rightarrow \;\; ae + b - \frac{1}{2} r a^2 \sigma^2 - c(e) \geq \underline{w} \\ IC \;\; &\Rightarrow \;\; e = \arg\max_{e'} \left\{ ae' + b - \frac{1}{2} r a^2 \sigma^2 - c\left(e'\right) \right\} \end{aligned}$$

- If cost is convex $\Rightarrow$ obj function is concave $\Rightarrow$ FOC is fine

  $\Leftrightarrow a = c'(e) \qquad$ IC

  $MB = MC$ (worker)

- Maximize principal's utility subject to constraint on agent's utility

- Raise b $\Rightarrow$ goes from principal's to agent's utility $\Rightarrow$ linear frontier

- Decompose Problem:

  - find best e to implement and corresponding a
    (how hard you work depends only on a, not b)

  - set b to satisfy IR constraints

  - since linear (transferable utility) $\Rightarrow$ max sum of utilities is same as maximizing one subject to another
    (make pie as big as possible subject to constraints)

– Principal's Objective: maximize total pie

- Total Certainty Equivalent (TCE)

$$\underbrace{e}_{\text{total output}} - \underbrace{c(e)}_{\text{cost of producing}} - \underbrace{\frac{1}{2} r a^2 \sigma^2}_{\text{risk premium}}$$

– Principal: maximize TCE subject to $a = c'(e) \Rightarrow$ substitute FOC

–

$$\max_e \; e - c(e) - \frac{1}{2} r \sigma^2 \left[c'(e)\right]^2$$

First-order condition:

$$1 - c'(e) - r^2 \sigma^2 \left[c''(e)\right] c'(e) = 0,$$

or,

$$c'(e) \left[1 + r\sigma^2 c''(e)\right] = 1.$$

Rearranging:

$$a = c'(e) \rightarrow a^* = \frac{1}{1 + r\sigma^2 c''(e)} \quad \text{(Incentive / Insurance tradeoff)}$$

– NOTES:

* If $r = 0$ or $\sigma^2 = 0 \Rightarrow$ sell the firm to the agent (e is FB), so that $c'(e) = 1, \; a^* = 1$
* Higher r, σ^2 : lower incentive-intensity.