# Comparative Statics for Mechanism Design

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- Question: Let $x \in \mathbb{R}$. When is $x^*(\theta) = \arg\max f(x, \theta)$ $\uparrow \theta$?

- Single Crossing Properties

  - *Definition:* $f(x, \theta)$ satisfies Single Crossing Property of Incremental Returns if $\forall \theta^H > \theta^L,\ \forall x^H > x^L$

$$f\left(x^H, \theta^L\right) - f\left(x^L, \theta^L\right) \geq (>) 0$$

$$\Rightarrow f\left(x^H, \theta^H\right) - f\left(x^L, \theta^H\right) \geq (>) 0.$$

  - Definition: If $f$ is differentiable in $x$, $f(x, \theta)$ satisfies Single Crossing Property of Marginal Returns (SCP-MR) if

$$\forall \theta^H > \theta^L,\ \forall x \quad f_1\left(x, \theta^L\right) \geq (>) 0 \Rightarrow f_1\left(x, \theta^H\right) \geq (>) 0.$$

- Discrete Case:

  - Suppose $S = \left\{x^L, x^H\right\}$ SCP-IR satisfied. Then, $x^*(\theta)$ is nondecreasing SSO.
  - $\theta < \theta',\ x^*(\theta) = \left\{x^L\right\}$
  - $\theta' \leq \theta \leq \theta'',\ x^*(\theta) = \left\{x^L, x^H\right\}$
  - $\theta > \theta'',\ x^*(\theta) = \left\{x^H\right\}$

- Theorem (Milgrom and Shannon):

  $x^*(\theta, S) = \arg\max_{x \in S} f(x, \theta)$ is nondecreasing $(SSO)$ in $(\theta, S)$

  IF AND ONLY IF

  $f$ satisfies SCP-IR.

  Note: necessity result relies on quantifying over sets.

- Definition: If $f(x, \theta)$ is differentiable in $x$, $f$ satisfies the Strict SCP-MR if:

$$\forall x,\ \forall\ \theta_H > \theta_L,\quad f_1\left(x, \theta^L\right) \geq 0$$

$$\Rightarrow \quad f_1\left(x, \theta^H\right) > 0$$

  Note: definition implies that agent can't be indifferent over region of $x$)

- Theorem: If $f$ is differentiable and $x^*(\theta)$ is interior, and if $f$ satisfies the Strict SCP-MR, then $x^*(\theta)$ is strictly increasing in $\theta$.

- Comparison:

  - $f$ satisfies the strong SCP-IR if, for all $\theta_H > \theta_L$ and all $x^H > x^L$,

$$f\left(x^H, \theta^L\right) - f\left(x^L, \theta^L\right) \geq 0$$

$$\Rightarrow f\left(x^H, \theta^H\right) - f\left(x^L, \theta^H\right) > 0$$

  - Fact: Strong SCP-IR not sufficient so that $x^*(\theta)$ strictly increasing in $\theta$.

- Only true if $f(x, \theta)$ is differentiable in $x$ and interior optimum.

- Mechanism Design
  - Agent's objective: $u(x, \theta) - t(x)$
  - Separation: $x^*(\theta)$ strictly increasing
  - Pooling: $x^*(\theta)$ constant somewhere

- Question: When is $x^*(\theta) = \arg\max_{x \in S} u(x, \theta) - t(x)$ nondecreasing for all $t(x)$?
  - Answer: necessary and sufficient condition is that $u$ is supermodular.
  - Motivation: $t(x)$ is endogenously determined. We would like comparative statics result to hold no matter what the principal chooses!
  - To see this: $t(x)$ is independent of $\theta$.
  - SCP-IR is not preserved by adding function $t(x)$
  - Sufficiency: supermodularity $\Rightarrow$ SCP-IR.
  - Necessity: $u(x, \theta) - t(x)$ satisfies SCP-IR for all $t$, iff $u$ is supermodular. See graph.

- General: $u(x, t(x), \theta)$ (assume $u$ differentiable, $u_2 \neq 0$)
  - Spence Mirrlees SCP (SM-SCP): $\frac{u_1}{|u_2|}(x, y, \theta) \uparrow \theta$
  - Fact: (Milgram/Shannon) $u(x, t(x), \theta)$ satisfies SCP-IR in $(x, \theta)$ $\forall t : \mathbb{R} \to \mathbb{R}$ if and only if $\frac{u_1}{|u_2|} \uparrow \theta$ (SM-SCP)
  - Same as SCP of $x - y$ indifference curves
  - higher types have steeper indifference curves

$$\begin{aligned} u(x, y, \theta) &= \bar{u} \\ u_1 dx + u_2 dy &= 0 \end{aligned}$$

$$\frac{dy}{dx} = -\frac{u_1}{u_2}$$

- Summary of Comparative Statics for Mechanism Design
  - Agent's objective: $u(x, t(x), \theta)$
  - If objective satisfies SCP-IR, $x^*(\theta)$ is nondecreasing in $\theta$ (SSO).
  - If objective satisfies strict SCP-IR, every selection from $x^*(\theta)$ is nondecreasing in $\theta$.
  - If obj. differentiable in $x$ and interior optimum, strict SCP-MR $\Rightarrow x^*(\theta)$ is strictly increasing.
  - In order to get "pooling," need non-differentiability.

- Generalized Envelope Theorem (Milgrom)

– Theorem: Let $S$ be a non-empty subset of a compact set and suppose that for all $\theta \in [\underline{\theta}, \bar{\theta}]$, $x^*(\theta) = \arg\max_{x \in S} f(x, \theta)$ is non-empty and $f(x, \theta)$ is continuous in $x$. Further assume that the partial derivative $f_\theta$ exists and is continuous in $(x, \theta)$. Then for any selection $\hat{x}(\theta)$ from $x^*(\theta)$,

$$f(\hat{x}(\theta), \theta) = f(\hat{x}(\underline{\theta}), \underline{\theta}) + \int_{\underline{\theta}}^{\theta} f_\theta(\hat{x}(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}.$$