3. Continuous Effort & First Order Approach (FOA)

- $\max_{e,s(-)} \int (x - s(x)) f(x \mid e)\, dx$ such that
  (IR) $\int u(s(x)) f(x \mid e)\, dx - c(e) \geq \underline{u}$
  (IC) $e \epsilon \arg\max_{e'} \int u(s(x)) f(x \mid e')\, dx - c(e')\,, \quad e \epsilon [\underline{e}, \overline{e}]$

  Remark 5 *In mechanism design, single crossing property. Incentive constraint → one eqn, but not true in moral hazard.*

- Strong Assumption: If A's objective is strictly concave in effort, then

$$IC \Leftrightarrow \frac{\partial}{\partial e} \int u(s(x)) f(x \mid e)\, dx = c'(e)$$

- <u>Question</u>: When is this integral concave in e?
  <u>Answer</u>: Depends on what we know about $u(s(x))$.

  - If $s'(x) > 0$, then since $u' > 0$, we conclude $u(s(x)) \uparrow x$.
    When is $\int g(x)\, dF(x \mid e)$ concave in e $\forall g$ nondecreasing $\left(\forall g \in G^{\text{nondecreasing}}\right)$?
  - Note:
    $G^{\text{nondecreasing}}$ is a closed, convex cone.
    Define $T^{\text{nondecreasing}} = \{t(x) = \mathbf{1}\{x \geq a\} \text{ for some } a \epsilon \mathbb{R}\}$
    <u>Fact</u>: closed, convex cone of $T^{\text{nondecreasing}} \cup \{1, -1\} = G^{\text{nondecreasing}}$

  - *Theorem:* $\int g(x)\, dF(x \mid e)$ is nondecreasing $\forall g \in G^{\text{nondecreasing}}$,
    if and only if $\int t(x)\, dF(x \mid e)$ nondecreasing in e $\forall t \in T^{\text{nondecreasing}}$
    if and only if $1 - F(x \mid e)$ nondecreasing in e $\forall x$

    But this also works for concavity :

– When is $\int g(x)\,dF(x\,|\,e)$ **concave**?
Theorem: $\int g(x)\,dF(x\,|\,e)$ is **concave** in e $\forall g \in G^{\text{nondecreasing}}$,
if and only if $1 - F(x\,|\,e)$ **concave** in e $\forall x$
if and only if $\int t(x)\,dF(x\,|\,e)$ **concave** in e $\forall t \epsilon T$

PROOF:
Use integration by parts argument or "convex combinations" method

– Note: approach also works for other sets of payoff functions. For example, for set of increasing concave functions, $G^{\text{nd-concave}}$, set of test functions is

$$T^{\text{nd-concave}} = \{t(x) = \min(x, a) \text{ for some } a \epsilon \mathbb{R}\}$$

Jewitt (1988) finds conditions under which $u(s(x))$ is nondecreasing and concave.

- Assumption: $F(x\,|\,e)$ is convex (CDF).
- CDF+$u(s(x))$ is nondecreasing in x, together imply that the agent's obj function is concave in effort.

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– CDF rarely satisfied. To see an example where it is:

$$F(x\,|\,e) = eF^H(x) + (1-e)F^L(x) \text{ for some } F^H \succeq_{MLR} F^L$$

– An example where we might question this assumption:
3 Outcomes:

$$(p_1(e), p_2(e), 1 - p_1(e) - p_2(e))$$

versus just choose points in the simplex:

$$(p_1, p_2, 1 - p_1 - p_2)$$

- Now, we show that CDF+MLRP implies that the FOA is valid. Proof approach:
  - First, assume CDF + MLRP.
  - Next, consider relaxed problem where only local constraints bind. Solve for optimal sharing rule for a given e.
  - Prove that under our assumptions, the solution to the relaxed problem, $\tilde{s}(x)$, is nondecreasing.
  - Observe that if $\tilde{s}(x)$ is nondecreasing, agent's objective is globally concave, and so only local constraints bind. Thus, $\tilde{s}(x)$ clears all IC constraints, local and global, and it is feasible in the original problem.
  - Since $\tilde{s}(x)$ solves a relaxed program and satisfies all constraints of the original program, it must be optimal.
  - Note: check corner solutions.
- Thus, it remains to show that if we assume CDF+MLRP (actually, CDF is not needed for this step), and solve the relaxed problem, where only local constraints bind, we get $\tilde{s}(x)$ nondecreasing.
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$$\text{Maximize}_{s(x)} \int (x - s(x)) f(x \mid e)\, dx$$

$$\text{such that} \int u(s(x)) f(x \mid e)\, dx - c(e) \geq \underline{u} \qquad \text{(IR-}\lambda\text{)}$$

$$\int u(s(x)) f_e(x \mid e)\, dx = c'(e) \qquad \text{(ICL-}\mu\text{)}$$

$$\text{FOC:} \quad \frac{1}{u'(s(x))} = \lambda + \mu \frac{f_e(x \mid e)}{f(x \mid e)}$$

Note: $\frac{f_e}{f} = \frac{\partial}{\partial e} \log(f(x \mid e))$

MLRP $\Leftrightarrow \frac{\partial^2}{\partial x \partial e} \log f \geq 0 \Leftrightarrow \frac{f_e}{f} \uparrow x \Leftrightarrow \frac{f(x|e_H)}{f(x|e_L)} \uparrow x$

Then, MLRP implies $\frac{f_e}{f}$ nondecreasing which in turn implies that the sharing rule is nondecreasing.