MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Economics

14.129 - Contract Theory, Fall 1999

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Solving Moral Hazard Models

1. Benchmark: effort is observable and contract can depend on it: $\tilde{s}(x, e)$
   - IC drops out.
     - Simply pay $\tilde{s}(x, e) = 0$ unless correct effort is chosen (depends on reservation utility)
   - First Best $\longrightarrow$ still use a sharing rule, $\tilde{s}(x, e) = s(x)$ if $e = e^*$
     - gains from trade in risk-sharing (unless same utility functions)
   - Program

$$\max_{e, s(\cdot)} \quad \int v\left(x - s\left(x\right)\right) f\left(x \mid e\right) dx$$

$$\text{s.t.} \quad \int u\left(s\left(x\right)\right) f\left(x \mid e\right) dx - c\left(e\right) \geq \underline{u} \qquad \text{(IR)}$$

1. •
   - λ multiplier on (IR)
   - Can drop IC due to observability, don't pay them unless see the effort specified

- To Solve: Take a 2 step approach
  - notice IR must bind (or can lower $u(s(x))$ by a constant, Principal better off
  - maximize the objective pointwise
    - if x is discrete, clearly this approach is right
    - optimal control: same thing works if you choose a function

- FOC: differentiate w.r.t. $s(x)$

$$-v'(x-s(x)) f(x \mid e) + \lambda u'(s(x)) f(x \mid e) = 0$$

$$\lambda = \frac{v'(x-s(x))}{u'(s(x))}, \quad \forall x$$

- Ratio of marginal utilities is constant (characteristic of optimal risk-sharing)
- Observe:
  - if $f(x \mid e)$ is degenerate $e \longrightarrow x_e$ (no uncertainty, no risk)
  - if $v(w) = w$ (risk neutral principal)
    - agent bears no risk
    - risk-neutral agent bears all the uncertainty (insurance contract)

$$\underbrace{\lambda}_{\text{optimal risk-sharing constant}} = \frac{1}{u'(s(x))} \Longrightarrow u'(s(x)) = \text{constant}$$
$$\Longrightarrow s(x) = \text{constant}$$

<u>Question</u>: If principal is risk-neutral, how do we find the effort?

Answer: IR

$$u\left(w\left(e\right)\right)-c\left(e\right)=\underline{u},\quad \text{(will bind)}$$

$$w\left(e\right)=u^{-1}\left[\underline{u}+c\left(e\right)\right]$$

– Principal (think of u^{-1} as converting monetary units to utils of principal)

$$\begin{aligned}&\max_{e}\underbrace{\int xf\left(x\left(e\right)\right)dx}_{\text{benefits}}-\underbrace{u^{-1}\left[\underline{u}+c\left(e\right)\right]}_{\text{cost}\longrightarrow\text{deterministic}}\\ =\;& E\left[x\mid e\right]-u^{-1}\left[\overline{u}+c\left(e\right)\right]\end{aligned}$$

– equalize marginal benefit (MB) of effort with monetary cost of effort

– To see this, take FOC⟶differentiate⟶get marginals⟶no distortion

2. Unobservable Effort

- No Risk Aversion (either principal or agent) $\longrightarrow$ no need for insurance

  Proposition 1 *First Best can be attained. Principal "sell the firm" to Agent.*

$$\begin{aligned} s(x) &= x - \alpha, \quad \alpha = \text{price of firm} \\ &\quad \text{(Agent is the residual claimant)} \end{aligned}$$

  IR: $\int xf(x \mid e)\,dx - c(e) - \alpha \geq \underline{u}$

  IC:

$$\begin{aligned} e^* &= \arg\max_e \int xf(x \mid e)\,dx - c(e) - \alpha \\ &= \arg\max_e E[\times \mid e] - c(e) \quad \text{(First Best incentives)} \\ &= \int v(x - s(x))\,f(x \mid e)\,dx + \int u(s(x))\,f(x \mid e)\,dx - c(e) \\ &\quad \text{(choice of effort that maximizes joint surplus)} \end{aligned}$$

  – $\longrightarrow e^*$ is FB
  set α so that IR binds (e.g. taxi pays dispatcher for the day and keeps the rest):

$$\alpha = E\left[x \mid e^{FB}\right] - c\left(e^{FB}\right) - \underline{u}$$

- Risk-Averse Agent, Principal Risk-Neutral (two effort model)

$$\begin{aligned} e_L, e_H \qquad c(e) &= c_H, c_L, \quad (c_H > c_L) \\ F(x \mid e) &= \begin{cases} F_H(x) \\ F_L(x) \end{cases} \end{aligned}$$

Remark 2 *Assume* $F_H(x) \underset{FOSD}{\geq} F_L(x)$

$\Longrightarrow F_H \leq F_L, \quad \forall x \quad \Leftrightarrow \quad \int g(x)\,dF_H(x) \geq \int g(x)\,dF_L(x), \quad \forall g$ *non-decreasing*

(check true for all step functions)

$E(g) = 1 - F \Longrightarrow$*increase mass to the right*

- Question: How is the e_L implemented?
  Answer: Optimal Risk Sharing
  - s_L is constant since $e_L > e_H \Longrightarrow IC$ satisfied
  - make IR bind with fixed wage (constant wage)

- Question: How is the e_H implemented?
  Answer:
  - sharing has to reward high output more

$$\max_{s(-)} \int (x - s(x)) f_H(x) \text{ such that}$$

$$IC : \int u(s(x)) f_H(x) - e_H \geq \int u(s(x)) f_L(x)\,dx - e_L, \quad \mu$$

$$IR : \int u(s(x)) f_H(x)\,dx - e_H \geq u, \quad \lambda$$

  - use pointwise optimization

$$\max_{s(x)} -s(x) f_H(x) + \mu\left[u(s(x))\left[f_H(x) - f_L(x)\right]\right] + \lambda u(s($$

$$\text{differentiate} : \quad -f_H(x) + \mu u'(s(x))(f_H(x) - f_L(x)) + \lambda u'(s(x)) f_H$$

$$\text{re-arrange} : \quad \frac{1}{u'(s_H(x))} = \lambda + \mu\left[1 - \frac{f_L}{f_H}\right]$$

- IR binds: if did not bind, sharing rule would not satisfy IC.

$$u(s(x)) - u(\hat{s}(x)) = \Delta \text{ doesn't depend on } x, \quad \lambda > 0$$

- Question: Could we have $\mu = 0$?
  Answer: full insurance, MU=constant
  - Contradiction - full insurance can't be incentive compatible (IC violated), $\lambda > 0$

- Interpretations
  - deviating from first best
    - not full insurance which we know is first best
    - μ: extent to which IC matters (distorts our choice)

  - Question: Given x we observe, how likely is it that they chose high effort?
    Answer: $\frac{f_H(x)}{f_L(x)}$ (likelihood ratio)
    - $\frac{f_H(x)}{f_L(x)} = 1 \longrightarrow$ don't want to reward or punish
    - low more likely than high $\longrightarrow$ punish
    - high more likely than low $\longrightarrow$ reward

- Optimal Sharing Rule varies only with $\frac{f_H(x)}{f_L(x)}$

$$\begin{aligned} \text{find } s^*(x) &= \frac{1}{u'(s^*(x))} = \lambda \\ s_H(x) &> s^*(x) \ iff \ \frac{f_H(x)}{f_L(x)} > 1 \\ s_H(x) &< s^*(x) \ iff \ \frac{f_H(x)}{f_L(x)} < 1 \end{aligned}$$

- Interpretation: rewarded for evidence of good behavior, punished for bad
  $s_H(x)$ strictly increasing in $\frac{f_H(x)}{f_L(x)}$

Remark 3 *This is not a statistical inference problem. In eqbm, you know what agent did (implement e_H). You behave as if inference problem. Provide incentives to get the other guy to truthfully reveal effort.*

- Question: $s_H(x)$ monotone in x?
  Answer: only true if $\frac{f_H(x)}{f_L(x)}$ is monotone
  - e.g.: suppose $x = e + \theta$, θ has two humps
    $s_H(x)$ monotone iff $\frac{f_H(x)}{f_L(x)} \uparrow$ (monotone) in x (MLRP)

- MLRP $\longrightarrow$ local. Can condition on two points.
  $MLRP \Longrightarrow FOSD$, but $FOSD \not\Leftarrow MLRP$

- PROOF: log supermodularity

$$\begin{aligned} f(x,e) &: \frac{\partial^2}{\partial x \partial e} f(x,e) \geq 0 \Leftrightarrow \text{supermodularity} \\ \frac{\partial^2}{\partial x \partial e} \ln f(x,e) &\geq 0 \Leftrightarrow \text{log supermodularity} \end{aligned}$$

- increasing differences: $f(x,e)$ supermodular $\Leftrightarrow$ $f(x,e_H) - f(x,e_L) \uparrow x \, \forall e_H > e_L$

- increasing ratios: $f(x,e)$ log supermodular $\Leftrightarrow$ $\frac{f(x,e_H)}{f(x,e_L)} \uparrow x \, \forall e_H > e_L$

Theorem 4 *$f(x,e)$ log sprm $\rightarrow$ $\int_a^b f(x,e)\,dx$ log sprm in (a,b,e) use for uncertainty, risk aversion, elasticity*

- f $MLRP$ $\Leftrightarrow$ f log sprm
  $\Longrightarrow \int_{-\infty}^{x} f(x,e)\,dx$ is log sprm $\Longrightarrow \frac{F(x,e_H)}{F(x,e_L)} \uparrow x \, \forall e_H > e_L$ (by def'n)
  but $\lim_{x \to \infty} \frac{F(x,e_H)}{F(x,e_L)} = 1$ (by def'n CDF)
  $\Longrightarrow F(x,e_H) \leq F(x,e_L)$

- x can be interpreted generally if Principal is risk neutral
  $\int (\pi - s(x)) f_H\left(\frac{\pi}{x}\right) dx$
  x is just a signal
  e.g.: $x =$ exam score, $\pi =$ how good a researcher you will be

- Add'l information - multiple signal

$$x = (x_1, x_2), \quad \frac{1}{u'(s_H(x_1, x_2))} = \lambda + \mu \left[1 - \frac{f_L(x_1, x_2)}{f_H(x_1, x_2)}\right]$$

$$s \text{ depends on } x_i \; iff \; \frac{f_H}{f_L}(x_1, ..., x_n) \text{ depends on } x_i$$

- if π is a sufficient statistic for $(x_1, ..., x_n)$ then WLOG $s(\pi)$

$$\frac{f_H(x_1, ..., x_n, \pi)}{f(x_1, ..., x_n, \pi)} = \frac{f_H(\pi)}{f_L(\pi)}$$

- Random Schemes are not optimal