MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Economics

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Professor Susan Athey | Phone: 3-6407

Office: E52-252C | Email: athey@mit.edu

Mechanism Design with Multiple Agents

- Setup
  - $i \in \{1, ..., I\}$
  - Each agent has private information, $\theta_i \in \ominus_i$

$$\ominus = \prod_{i=1}^{I} \ominus_i$$

$$\theta = (\theta_1, ..., \theta_I)$$

  - y is an allocation, if

$$\begin{aligned} y &= (x, t) \\ x &= (x_1, ..., x_I) \\ t &= (t_1, ..., t_I) \end{aligned}$$

  - P: chooses y; has utility $V(y, \theta)$

- A_i: has utility $u_i(y, \theta)$
- Joint distribution of types $F(\theta)$
  - $F(\theta_{-i} \,|\, \theta_i)$ conditional distributions
  - Assume that conditional distributions have strictly positive densities

- Mechanism:

$$\Gamma^c = \left\{ \begin{array}{c} \mu, \ominus, F, \; y : \mu \rightarrow \mathbb{R}^I \times \mathbb{R}^I, \\ u_i : \mathbb{R}^I \times \mathbb{R}^I \times \ominus \rightarrow \mathbb{R}, \; i = 1, ..., I \end{array} \right\}$$

Γ^c communication game

$\mu = (\mu_1, ..., \mu_n)$ space of messages (action space)

$\ominus$ types

F beliefs

$y(m)$ contract

(x, t) allocation

$u_i(x, t, \theta)$ utilities

- Game:
  - P offers Γ^c
  - A chooses whether to participate
  - Agents that accept play Γ^c
    i.e. report $m_i \in \mu_i$ and receive payoffs $u_i(y(m), \theta)$
  - Solution concept: Bayesian - Nash equilibrium.
  - $m^*(\theta) = (m_1^*(\theta), ..., m_I^*(\theta_I))$ is an equilibrium if, for all i and all $m_i' \in \mu_i$,

$$E_{\theta_{-i}} \left[u_i(y(m^*(\theta)), \theta) \,|\, \theta_i \right] \geq E_{\theta_{-i}}[u_i(y(m_i', m^*(\theta_{-i})), \theta) | \theta_i]$$

- Examples:
  - Screening
  - Public goods problem: build a bridge
    $x_i = x_j \, \forall \, i, j$
    θ_i how much you want it
  - Bargaining
    - x_1 produced by seller
    - x_2 consumed by buyer
    - $x_1 = x_2$
    - $-t_1 = t_2 =$ price
    - θ_1 cost
    - θ_2 value to a buyer

- Revelation Principle:
  - Consider a communciation game Γ^c.
  - Let

$$m^*(\theta) = (m_1^*(\theta), ..., m_I^*(\theta_I))$$

    be a BNE of game (w.l.o.g all participate).

  - Then $\exists$ another communication game

$$\Gamma^d = \left\{ \begin{array}{c} \mu = \ominus, \ominus, F, \ \tilde{y} : \ominus \to \mathbb{R}^I \times \mathbb{R}^I, \\ u_i : \mathbb{R}^I \times \mathbb{R}^I \times \ominus \to \mathbb{R}, \ i = 1, ..., I \end{array} \right\}$$

    where

$$\widetilde{y}(\widehat{\theta}) = y(m^*(\widehat{\theta})),$$

    and such that there exists a BNE of Γ^d satisfying

$$m_i(\theta_i) = \theta_i.$$

– Proof:

  * Check inequalities for BNE.

$$\begin{array}{rcl} E_{\theta_{-i}}\left[u_i(y(m^*(\theta)),\theta)\,|\,\theta_i\right] & \geq & E_{\theta_{-i}}[u_i(y(m_i', m^*(\theta_{-i})),\theta)|\theta_i] \quad \forall m_i' \\ & \Rightarrow & \\ E_{\theta_{-i}}\left[u_i(\tilde{y}(\theta),\theta)\,|\,\theta_i\right] & \geq & E_{\theta_{-i}}[u_i(\tilde{y}(\hat{\theta}_i,\theta_{-i}),\theta)|\theta_i] \quad \forall \hat{\theta}_i \end{array}$$

- Does the Revelation Principle imply that the principal can separate types?

  – No, since if m is the original equilibrium and

$$y_i(m_i(\theta_i),\, m_i(\theta_i)) = y_i(m_i(\theta_i'),\, m_{-i}(\theta_{-i})),$$

  – then

$$\widetilde{y}_i(\theta_i,\theta_{-i}) = \widetilde{y}_i(\theta_i',\theta_{-i}).$$

  – Hence, if initial equilibrium did not separate types, the corresponding direct revelation mechanism does not either.

- Two different solution concepts in multiplayer mechanism design: BNE and Dominant Strategy Equilibrium
  - BIC (Bayesian Incentive Compatibility Constraint)

    $$E_{\theta_{-i}}\left[u_i(y(\theta_i, \theta_{-i}), \theta) \,|\, \theta\right] \geq E_{\theta_{-i}}[u_i(y(\widehat{\theta}_i, \theta_{-i}), \theta_i)|\theta] \quad \forall \theta_i, \widehat{\theta}_i$$

    * $\widetilde{m}(\theta_i) = \theta_i$ in Γ^d is a BNE $\Leftrightarrow$ "truthtelling satisfies BIC"

  - DIC (Dominant Strategy Incentive Compatibility Constraint)

    $$u_i(y(\theta_i, \theta_{-i}), \theta) \geq u_i(y(\widehat{\theta}_i, \theta_{-i}), \theta)$$

    $$\forall \theta, \widehat{\theta}_i$$

    DIC requires BIC to hold pointwise

  - There exists an analogous revelation principle for dominant strategy implementation.

Mechanism Design with Multiple Agents and Quasi-Linear Preferences

- Consider the special case with quasi-linear preferences,

$$U^i(\widehat{\theta} \mid \theta) = x_i(\widehat{\theta})\, \theta_i - t_i(\widehat{\theta}),$$

and independent types:

$$\bar{U}_i(\widehat{\theta}_i \mid \theta_i) = \overline{x}_i(\widehat{\theta}_i)\theta_i - \overline{t}_i(\widehat{\theta}_i)$$

$$\bar{x}_i(\theta_i) = E_{\theta_{-i}}(x_i(\theta))$$

$$\overline{t}_i(\theta_i) = E_{\theta_{-i}}(t_i(\theta))$$

(Note: preferences above satisfy SCP)

- Generalized constraint reduction theorem

SCP holds with quasilinear preferences automatically

 – (x, t) is BIC $\Leftrightarrow$
 $\forall i \;\; \overline{x}_i\,(\theta_i)$ is nondecreasing and

$$\frac{d}{d\theta_i}\overline{U}^i(\theta_i \mid \theta_i) = \overline{U}^i_2(\theta_i \mid \theta_i) \text{ a.e.}$$

 – Further consequence:

$$\begin{aligned}\overline{U}^i(\theta_i \mid \theta_i) &= \overline{U}^i(\underline{\theta}_i \mid \underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{U}^i_2(s \mid s) ds \\ &= \overline{U}^i(\underline{\theta}_i \mid \underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{x}_i\,(s) ds\end{aligned}$$

– For transfers:

$$\overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i) = \overline{x}_i(\underline{\theta}_i)\underline{\theta}_i - \overline{t}_i(\underline{\theta}_i)$$

For a given $x(\cdot)$, $\overline{t}_i(\underline{\theta}_i)$ is determined by $\overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i)$

– If

$$\overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i) = 0$$

Then

$$\overline{t}_i(\underline{\theta}_i) = \underline{\theta}_i \overline{x}_i(\underline{\theta}_i)$$

– In general $\overline{U}^i(\underline{\theta}_i \,|\, \theta_i)$ determines the intercept of $\overline{t}_i(\theta_i)$

– BIC, in turn, determines "slope" of $\overline{t}_i(\theta_i)$

$$\frac{d}{d\theta_i} U^i(\theta_i \,|\, \theta_i) \underset{a.e.}{=} \underbrace{\overline{x}_i'(\theta_i)\theta_i - \overline{t}'(\theta_i)}_{=0 \text{ (f.o.c.)}} + \overline{x}_i(\theta_i) = \underbrace{\overline{x}_i(\theta_i)}_{\text{slope for } \overline{t}(\theta_i)}$$

Together, $\overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i)$ and $x(\theta)$ BIC determine unique $\overline{t}_i(\theta_i)$

– Note: $\overline{t}_i(\theta_i)$ are only expected transfer functions. There are a lot of mechanisms that result in the same $\overline{t}_i(\theta_i)$

- Generalized Revenue Equivalence Theorem

  – Suppose (x^D, t^D) is DIC.

  – Suppose further $\exists$ (x^B, t^B) s.t.:

$$(1) \quad (x^B, t^B) \text{ is BIC}$$

$$(2) \; \bar{x}^{iB}(\theta_i) = E_{\theta_{-i}}[x_i^D(\theta_i, \theta_{-i})]$$

$$(3) \quad E_{\theta_{-i}}[t_i^D(\underline{\theta}_i, \theta_{-i})] = \overline{t}_i^B(\underline{\theta}_i) = E_{\theta_{-i}}[t_i^B(\underline{\theta}_i, \theta_{-i})]$$

[(3) implies that expected utility to $\underline{\theta}_i$ is the same under both mechanisms]

– Then

$$E_{\theta_{-i}}[t_i^D(\theta_i, \theta_{-i})] = \overline{t}_i^B(\theta_i) = E_{\theta_{-i}}[t_i^B(\theta_i, \theta_{-i})]$$

and thus

$$\bar{U}^{iB}(\theta_i \,|\, \theta_i) = \bar{U}^{iD}(\theta_i \,|\, \theta_i)$$

- Proof:
  – By the constraint reduction theorem, $\overline{t}^{iB}(\theta_i)$ is uniquely determined by $\bar{x}^{iB}(\theta_i)$ and $\overline{t}_i^{iB}(\underline{\theta}_i)$, since

$$\begin{aligned} \overline{U}^i(\theta_i \,|\, \theta_i) &= \overline{x}_i(\theta_i)\theta_i - \overline{t}_i(\theta_i) \\ &= \overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{x}_i\,(s)ds. \end{aligned}$$

  – Let

$$\overline{t}_i^B(\theta_i) = E_{\theta_{-i}}[t_i^D(\theta_i, \theta_{-i})]$$

  – Know that (x^B, t^B) is BIC, since (x^D, t^D) is DIC.
  – Finally,

$$\overline{t}^D(\theta_i) = \overline{t}^B(\theta_i), \text{ same } \overline{x}^D(\theta_i) = \overline{x}^B(\theta_i)$$

$$\Rightarrow \quad U^{iB}(\theta_i \,|\, \theta_i) = E_{\theta_{-i}}[U^{iD}(\theta \,|\, \theta)]$$

- Example:

  Auctioning: single unit

$$x_i(\theta) \in [0,1]$$

$$\sum_{i=0}^{I} x_i(\theta) = 1$$

  $x_0(\theta)$ probability principal keeps object

  $x_i(\theta)$ probability agent i gets the object

  - 2nd price auction

$$x_i(\theta) = 1 \text{ iff } \theta_i > \theta_j \ \ \forall j \neq i$$

$$t_i(\theta) = \begin{cases} \max\limits_{i \neq j}\{\theta_j\} \text{ if } x_i(\theta)=1 \\ 0 \text{ otherwise} \end{cases}$$

  - "Indirect" implementation:
    * Each player makes a bid $\beta_i^2(\theta_i)$
    * Win if: $\beta_i^2(\theta_i) > \beta_j^2(\theta_j) \ \ \forall j \neq i$
    * If win, pay:

$$\max_{i \neq j}\{\beta_j^2(\theta_j)\}$$

    * Claim: $\beta_i^2(\theta_i) = \theta_i$ is a dominant strategy
    * Pf: "pivotal mechanism," different bid only changes payment if it changes allocation.

  - 1st Price Auction:

- Each player makes a bid $\beta_i^1(\theta_i)$
- Win if:

$$\beta_i^1(\theta_i) > \beta^1(\theta_j) \ \ \forall j$$

- If win, pay β^1.
- Bid solves:

$$\max_b (\theta_i - b) \ \ \text{Prob}(b \text{ wins})$$

- Direct approach: write down a set of differential equations and solve.

– Observations:

- Suppose $\exists$ a symmetric BNE. Then by the revelation principle $\exists$ a direct revelation mechanism which is BIC and implements the same allocation.
- For the second price auction the highest type wins. If 1st price auction is symmetric and has strictly increasing strategies (which will be true in equilibrium) then allocation is the same as in 2nd price.
- Assuming atomless type distribution, expected value of $\underline{\theta}_i$'s payment in both first and second price auction is zero.

– Conclude:

- First and second price auctions have the same expected transfers and same expected utilities.
- The conclusion follows from the Generalized RET.

• Revenue Equivalence Theorem:

– Informally: Any auction with the same allocation rule and same expected utility to $\underline{\theta}_i$ has same expected utility for each agent and same expected revenue to the auctioneer.

– Implication: Bid in 1st Price Auction when bidders are symmetric:

$$\beta_i^1(\theta_i) = E[\max_{i \neq j} \beta_j^2(\theta_i) \,|\, \theta_i > \theta_j \;\; \forall\, j] = E[\max_{i \neq j} \theta_j \,|\, \theta_i > \theta_j \;\; \forall j]$$

(Note: probability of winning is the same in each auction, so that consider only payment conditional on winning).

– Now, formalize.

- Theorem:

  – Consider a private-values auction environment with I risk-neutral bidders and independently distributed valuations (that is, $\overline{U}^i(\theta_i \,|\, \theta_i) = \bar{x}_i(\theta_i)\theta_i - \bar{t}_i(\theta_i)$). Suppose two different auction procedures have BNE that satisfy:

$$\text{(i)} \;\; \overline{x}_i(\theta_i) \text{ is the same}$$

$$\text{(ii)} \;\; \bar{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i) \text{ is the same}$$

  – Then, auctions generate same expected payoff to each participant, and thus to the auctioneer.

- Proof:

$$\overline{U}^i(\theta_i \,|\, \theta_i) = \overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{x}_i(s)\, ds$$

  In the case of symmetric bidders and symmetric strategies:

$$\overline{x}_i(s) = \Pr(\theta_j < s \;\; \forall\, j \neq i) = [F(s)]^{n-1}$$

- Theorem depends on:

- Quasilinear preferences (risk-neutral)
  Suppose utility given by $V^i(x_i, t_i, \theta_i)$, in this case U_2 will depend on transfer, that is

$$\overline{U}^i(\theta_i \,|\, \theta_i) = \overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} E_{\theta_{-i}}[V^i_{\theta_i}((x_i, t_i)(s, \theta_{-i}), s)]\, ds$$

  * With risk-averse bidders FPA generates more revenue since bidders are reluctant to take a gamble of losing the bid

- If types correlated, get as integrand

$$\begin{aligned} &E_{\theta_{-i}}[V^i_{\theta_i}((x_i, t_i)(s, \theta_{-i}), s) \mid s] \\ &+ \frac{\partial}{\partial z} E_{\theta_{-i}}[V^i((x_i, t_i)(s, \theta_{-i}), s) \mid z] \Big|_{z=s} \end{aligned}$$

- R.E.T.$\Rightarrow$ transfers are the same in expectation.

  Take 2nd price auction (SPA):

$$t_i^2(\theta) = \mathbf{1}_{\theta_i > \max_{j \neq i} \theta_i} \bullet \max_{j \neq i} \theta_j$$

$$E_{\theta_{-i}}[t_i^2(\theta)] = \Pr(\theta_i > \max_{j \neq i} \theta_i) \bullet E(\max_j \theta_j \,|\, \theta_i > \max_{j \neq i} \theta_i)$$

$$E_{\theta_{-i}}[t_i^1(\theta)] = \Pr(\theta_i > \max_{j \neq i} \theta_i) \bullet \beta_i^1(\theta_i)$$

  So,

$$\beta_i^1(\theta_i) = E(\max_j \theta_j \,|\, \theta_i > \max_{j \neq i} \theta_i)$$

  - Ex: $\theta_i \tilde{} U[0, 1]$, two firms
    * Computations from 2nd price
    * For θ_i, probability of winning is θ_i, expected payment conditional on winning is $\theta_i/2$
    * Optimal bid in 1st price for type θ_i must then be $\theta_i/2$.

Optimal Auctions

- θ_0 principal's value of the object
  - Principal's surplus is:

$$\pi = E[\underbrace{(1 - \sum_{i=1}^{I} x_i(\theta))\theta_0}_{\text{value of object to } P} + \underbrace{\sum x_i(\theta)\,\theta_i - \sum_{i=1}^{I} \overline{U}^i\,(\theta_i \,|\, \theta_i)}_{t_i}]$$

  - If BIC holds:

$$= E\left[(1 - \sum_{i=1}^{I} x_i(\theta))\theta_0 + \sum_{i=1}^{I} x_i(\theta)\left[\theta_i - \underbrace{\frac{1 - F_i(\theta_i)}{f_i(\theta_i)}}_{\text{info. rent}}\right] - \sum_{i=1}^{I} \bar{U}^i\,(\underline{\theta}_i \,|\, \underline{\theta}_i)\right]$$

  - Define virtual type:
    produces surplus, creates incentive problems for higher types

$$J_i(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}$$

$$\pi = E\left[\sum_{i=1}^{I} x_i(\theta)[J_i(\theta_i) - \theta_0] - \sum_{i=1}^{I} \overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i)\right]$$

  - If IR binds to zero:

$$\overline{U}^i(\underline{\theta}_i \,|\, \underline{\theta}_i) = 0$$

– Solve:

$$\max \sum_{i=1}^{I} x_i(\theta)[J_i(\theta_i) - \theta_0] \text{ s.t. } \sum_{i=1}^{I} x_i(\theta) = 1$$

$$x_i(\theta) = \begin{cases} 1 \text{ if } J_i(\theta_i) > J_k(\theta_k) \text{ and } J_i(\theta_i) > \theta_0 \ \ \forall\, k \neq i \\ 0 \text{ otherwise} \end{cases}$$

* If types are symmetric ($F_i = F$) and $\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$ is $\nearrow \theta_i$, then

$$J_i(\theta_i) > J_k(\theta_k) \Leftrightarrow \theta_i > \theta_k$$

$\Rightarrow$ optimal auction entails efficient allocation among bidders.

* If MHC holds, then $J_i(\theta_i) \nearrow$
* Reserve prices:
  Optimal rule: sell the object iff:

$$\theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)} > \theta_0$$

We could have $\theta_i > \theta_0$ but $J(\theta_i) < \theta_0$, in which case the prinicpal withholds good inefficiently

- Asymmetric Bidders
  - Bulow and Roberts, JPE, 1988: Auctions as monopoly pricing
  - If $F_i \neq F_K$, then $J_i(\theta_i) > J_k(\theta_k)$ is not the same as $\theta_i > \theta_k$
  - Bias against strong types.

Efficient Mechanisms

- Setup
  - In auctions: second price auction is (DIC), use it to construct transfers and utility in a (BIC) mechanism
  - The same logic applies to more general context: 2nd price aution could be generalized to Groves mechanism (Vickery/Clark).
    * Agent preferences:

$$u^i(x_i, \theta_i) - t_i$$

    * x project choice, public good.

  - Definition:
    Allocation $\{x_i^*(\theta)\}_{i=1}^I$ is efficient if and only if, for all $\widetilde{x}$,

$$\sum_{i=1}^{I} u^i(x_i^*(\theta), \theta_i) \geq \sum_{i=1}^{I} u^i(\widetilde{x}_i, \theta_i)$$

    * Could we implement efficient allocation in dominant strategies? Answer is given by Vickery-Groves-Clark mechanism.

- Vickrey-Groves-Clark:
  - <u>Theorem:</u>
    * Let $x^*(\theta)$ be efficient.
    * Then (x^*, t^*) is DIC if, for $i = 1, ..., I$,

$$t_i(\theta) = -\sum_{j \neq i} u^j(x_j^*(\theta), \theta_j) + h_i(\theta_{-i}) \text{ for some } h_i(\theta_{-i}).$$

– Proof:

$$\begin{aligned} u^i(\theta_i \,|\, \theta) &= u^i(x_i^*(\theta), \theta_i) - t_i \\ &= u^i(x_i^*(\theta), \theta_i) + \sum_{j \neq i} u^j(x_j^*(\theta), \theta_i) - h_i(\theta_{-i}) \\ &= \sum_j u^j(x_j(\theta), \theta_j) - h_i(\theta_{-i}) \end{aligned}$$

Objective of each agent is social objective.

$$u^i(\widehat{\theta}_i \,|\, \theta) = \sum_{j=1}^{I} u^j(x_j^*(\widehat{\theta}_i, \theta_{-i}), \theta_j) - h_i(\theta_{-i})$$

– Can we find $h_i(\theta_{-i})$ to balance budget:

$$\sum_{i=1}^{I} t_i(\theta) = 0$$

No, in general.

- Special version: pivotal mechanism (Clark)
  – Define:

$$x^{-i*} = \arg\max \sum_{j \neq i} u^j(x, \theta_i)$$

Optimal allocation in the world where agent i does not exist

  – Define:

$$h_i(\theta_{-i}) = \sum_{j \neq i} u^j(x^{-i*}(\theta_{-i}), \theta_j)$$

  – Example: 2nd price auction. Assume θ_i is highest. Then:

$$\sum_{j \neq i} u^j(x^{-i*}(\theta_{-i}), \theta_j) \;=\; \max_{j \neq i} \theta_j$$

$$
\begin{aligned}
t_i(\theta_i) &= -\sum_{j \neq i} u^j(x_j^*(\theta), \theta_j) + h_i(\theta_{-i}) \\
&= -0 + \max_{j \neq i} \theta_j \\
&= \max_{j \neq i} \theta_j
\end{aligned}
$$

- Interpretation: if you are pivotal, you pay your externality on the world
- Conclusion:
  DIC can implement efficient allocation, but budget is not balanced

- Bayesian implementation:
  - Can get balanced budget by using "expected externality" mechanism if there are no IR constraints.
  - How to make transfers add up?
  - Define:

$$
t_i(\theta) = \underbrace{-E_{\theta_{-i}}[\sum_{i \neq j} u^j(x^*(\theta_i, \theta_{-i}), \theta_j)]}_{g_i(\theta_i)} + h_i(\theta_{-i})
$$

  - BB:

$$
\sum_{i=1}^{I} t_i(\theta) = 0 \Rightarrow \sum_{i=1}^{I} [h_i(\theta_{-i}) - g_i(\theta_i)] = 0
$$

  - Let:

$$
h_i(\theta_{-i}) = \frac{1}{I-1} \sum_{i \neq j} g_j(\theta_j)
$$

  - Problem: the participation constraint might be violated. After observing type, type θ_i might want to back out. But, incorporating IR constraints would destroy BB

- Example: Myerson-Satterthwaite Theorem
  - Bilateral Bargaining
  - Thm: Cannot have efficient allocation, BIC, budget-balance, and participation constraints.
  - Problem Set: Work through proof using techniques outlined above.
    * Construct appropriate pivotal mechanism in dominant strategies
    * Construct BIC transfers that satisfy budget-balance, taking expectation of DIC mechanism and adjusting as above
    * Construct BIC transfers that satisfy participation constraints, taking expectation of DIC mechanism
    * Use properties of DIC mechanism (that hold pointwise) to prove theorem
    * Note: this proof approach differs from most texts and from original paper!