Linear Contracting Models with Multiple Efforts

- Multiple tasks $e_1 ... e_{n.}$
  - Output:

$$\begin{aligned} x_i &= e_i + \theta_i \\ \theta &\sim N(0, \Sigma) \end{aligned}$$

    Σ allows for correlation between tasks

  - Cost:

$$c(e) = c(e_1, ..., e_n)$$

  - Principal has some benefit function $B(e)$.
  - Contracts:

$$s(x) = \alpha x + \beta$$

  - Note: CE in normal exp. model is

$$\alpha e + \beta - \frac{1}{2} r \, \alpha' \Sigma \alpha - c(e)$$

    β shifts surplus $\Rightarrow$ linear Pareto frontier

  - TCE:

$$B(e) - c(e) - \frac{1}{2} r \, \alpha' \Sigma \alpha$$

$$\max_{\alpha} \quad B(e) - c(e) - \frac{1}{2} r \, \alpha' \Sigma \alpha$$

$$\text{s.t. } \alpha = \nabla_e c(e) \tag{IC}$$

$$\text{i.e., } \frac{\partial c}{\partial e_i} = \alpha_i.$$

Observe:

$$D\alpha(e) = D^2 c(e) = \begin{bmatrix} \frac{\partial^2}{\partial e_1^2} c(e) & \cdots & \frac{\partial^2}{\partial e_1 \partial e_n} c(e) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial e_n \partial e_1} c(e) & \cdots & \frac{\partial^2}{\partial e_n^2} c(e) \end{bmatrix}$$

FOC:

$$\begin{aligned} \nabla B(e) &= \nabla c(e) + r\, D^2 c(e)\, \Sigma\, \nabla c(e) \\ &= (I + r\, D^2 c(e)\, \Sigma)\, \nabla c(e) \\ \alpha^* &= (I + r\, D^2 c(e)\, \Sigma)^{-1} \nabla B(e) \end{aligned}$$

- Case 1:
  - Outputs stochastically independent, efforts independent in cost function.

$$\begin{aligned} \Sigma_{ij} &= 0, \quad \text{for } i \neq j \\ \frac{\partial^2 c}{\partial e_i \partial e_j} &= 0, \quad \text{for } i \neq j \end{aligned}$$

  - This yields

$$\alpha_i = \frac{\partial B / \partial e_i}{1 + r \frac{\partial^2 c}{(\partial e_i)^2} \sigma_i^2}$$

- Case 2:
  - Unmonitored tasks: two tasks ($c(x)$ convex)

$$\sigma_{12} = 0;\ B_{12} = 0;\ c_{12} = 0;\ \sigma_2^2 \to \infty$$

$$\alpha_1 = \frac{B_1 - \frac{c_{12}}{c_{22}} B_2}{1 + r\sigma_1^2 (c_{11} - \frac{c_{12}^2}{c_{22}})}$$

$$(c_{11} - \frac{c_{12}^2}{c_{22}}) > 0 \quad \text{(convexity)}$$

- $\alpha_2 = 0 \;\; \rightarrow$ no incentive for task 2 (too expensive)
- If $c_{12} > 0$, $B_2 > 0$, then $\alpha_1 \downarrow$
- Incentives to activity 1 are also decreasing
- Suppose:

$$\begin{aligned} c(e) &= \hat{c}(e_1 + e_2) \\ B(e_{1,} 0) &= 0 \quad \forall e_1 \end{aligned}$$

  Then, as $\sigma_2^2 \rightarrow \infty$, $\alpha_2 \rightarrow 0$.
- If $\alpha_1 > 0$, then $e_2 = 0$, leading to no benefit to the principal ($B = 0$).
- Hence if $\alpha = (0, 0)$ induces any effort it is better.
- Justification for low-powered incentives!

• Holmstrom and Milgrom, AER 1994: "Firm as an Incentive System"

  - Suppose task 2 is asset maintenance (machines, firm reputation)
  - Solution: $\alpha_2 = 1$ using ownership
  - Two systems:
    * high-powered incentives, ownership
    * "employee"–firm owns assets, fixed wages
  - Under what conditions are $\alpha_1,\ \alpha_2$ complements in the principal's objective function?

$$TCE = TCE(\alpha_1, \alpha_2, \sigma_1^2, \sigma_2^2, \sigma_{12}, r)$$

- Assumptions on the cost function: assume quadratic for simplicity.
  - Efforts substitutes in the cost function: $c_{12}(e_1, e_2) \geq 0$.
    Implies $\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} TCE \geq 0$ if c is quadratic.
  - Efforts complements in the cost function: $c_{12}(e_1, e_2) \leq 0$.
    Implies $\frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} TCE \leq 0$ if c is quadratic.

- Examples of questions to ask in this framework:
  - What happens when σ_1^2 changes?
  - How does the choice of $\alpha_1, \alpha_2, \sigma_1^2$ vary with the cost of changing technology?

- Intuition: when efforts are substitutes in the cost function, incentive instruments are complements to the principal.

Comparative Statics Detour

Comparative statics tools: Need for firm as a system and for mechanism design, as well as Hart-Moore 1990.

$$x^*(\theta, S) = \arg\max_{x \in S} f(x, \theta)$$

- When is $x^*(\theta, S)$ increasing in θ?

  Question: Does a solution exist? Are there multiple optima?

  $x^*(\theta, S)$ could be empty or a set

- What is an order?
  - partial order: $x, y \in X$, could have $x \not\geq y$ *and* $y \not\geq x$
  - Componentwise Order on $\mathbb{R}^n$: $a \geq b \; iff \; a_i \geq b_i; \; \forall i = 1, ..., n$
    - Notation: Meet and Join

$$a \vee b = (\max(a_1, b_1), ..., \max(a_n, b_n))$$

$$a \wedge b = (\min(a_1, b_1), ..., \min(a_n, b_n))$$

  - versus complete order
    - e.g., lexicographic order, usual order on real line
  - Orders Over Sets
    - Inclusion order: $A \geq_I B \; if \; B \subseteq A$

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    - Strong Set Order, Case 1: $A, B \subset \Re$
      - $A \geq_{SSO} B \Leftrightarrow$
        $a \in A \;\; and \;\; b \in B \quad \Rightarrow \quad \max(a,b) \in A, \;\; \min(a,b) \in B$
      - Equivalent definition: $x \in B \backslash A, \;\; y \in A \cap B, \;\; z \in A \backslash B \Rightarrow x \leq y \leq z$
    - Case 2: $A, B \subset \Re^n$
      - $A \geq_{SSO} B \Leftrightarrow$
        $a \in A \;\; and \;\; b \in B \quad \Rightarrow \quad a \vee b \in A, \;\; a \wedge b \in B$
    - Fact: $x^*(\theta) \uparrow \theta \; in \; SSO$

$$\Rightarrow \qquad \underbrace{x^H(\theta)}_{\text{highest member of } x^*(\theta)} \qquad , \qquad \underbrace{x^L(\theta)}_{\text{lowest member of } x^*(\theta)} \qquad \uparrow \theta$$

- General Definitions:
  - Lattice: set X, partial order $\geq$
  - "meet" $\wedge$, "join" $\vee$
    $x \vee y = \inf\{z : z \geq x, \; z \geq y\}$
    $x \wedge y = \sup\{z : z \leq x, \; z \leq y\}$
  - Sublattice: $S \subseteq X$ such that $x \in S, \; y \in S$
    $\Rightarrow x \vee y \in S \; and \; x \wedge y \in S$
  - Supermodularity: $f : X \Rightarrow \mathbb{R}$ is SPM, if $\forall x, y \in X$,

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$$

$$\Leftrightarrow \quad f(x \vee y) - f(x \wedge y) \geq [f(x) - f(x \wedge y)] + [f(y) - f(x \wedge y)]$$

  - Suppose $x = \left(z_1^H, z_2^L\right), \; y = \left(z_1^L, z_2^H\right)$

$$x \vee y = \left(z_1^H, z_2^H\right), \qquad x \wedge y = \left(z_1^L, z_2^L\right)$$

Then,

$$f(x \vee y) - f(x) \geq f(y) - f(x \wedge y)$$

$$\Leftrightarrow$$

$$f\left(z_1^H, z_2^H\right) - f\left(z_1^H, z_2^L\right) \geq f\left(z_1^L, z_2^H\right) - f\left(z_1^L, z_2^L\right)$$

- Restated question: When is $x^*(\theta, S)$ increasing in θ in strong set order?
  - Question: What if $x^*(\theta) = \emptyset$ for some θ?
    Fact: $\emptyset \geq_{SSO} A \geq_{SSO} \emptyset \ \forall A$.

- Theorem (Topkis, 1978): If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, it is SPM if $\forall i \neq j$, $\forall x$

  $$\frac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0.$$

  Proof: Apply definition of derivative.

- Interpretation:
  - Increasing first component increases marginal returns to other components
  - "Supermodular" $\Leftrightarrow$ for product set, every pair of choices complements

- Theorem (Topkis, 1978): If $f(x, \theta)$ is SPM and S is a sublattice, then $x^*(\theta, S) = \arg\max_{x \in S} f(x, \theta)$ is nondecreasing in θ in the strong set order.

SPM is not necessary. We will discuss necessity later.

- Games:

  Player payoffs: $f^1(x_1, x_2, \theta_1)$; $f^2(x_1, x_2, \theta_2)$

  Suppose for each i, f^i SPM in (x_1, x_2)

  and, f^i is supermodular in components of x_i if x_i is a vector.

- Theorem (Topkis, 1979): If each f^i is SPM in (x_i, x_j, θ), then eqm $(x_1^*, x_2^*) \uparrow \theta$.

- Applying the theorem:
  - use θ to parametrize diff't games

    $\theta_L: \ f^i(x_1, x_2, \theta_L) = f(x_i, x_j) - c^i(x_i) - c^j(x_i)$

    $\theta_H: \ f^i(x_1, x_2, \theta_H) = f(x_i, x_j) - c^i(x_i)$ (don't experience cost on other player)
  - Social θ_L versus Nash Equilibrium θ_H

    if $c^i, c^j \uparrow x$ then game has "too much" x
  - See Ilya Segal, "Contracting with Externalities."

Applying Comparative Statics to Multitask Problems

- Example: the two-effort linear contracting model.

  The risk premium is:

  $$-\frac{1}{2}r\alpha_1^2\sigma_1^2 - \frac{1}{2}r\alpha_2^2\sigma_2^2 - r\alpha_1\alpha_2\sigma_{12}$$

  Verify that when c quadratic, $c_{12} \geq 0$, we have:

  $$\frac{\partial^2}{\partial\alpha_1\partial\alpha_2} TCE\left(\alpha_1, \alpha_2, \sigma_i^2\right) \geq 0$$

Verify: $\frac{\partial^2}{\partial\alpha_1\partial\sigma_1^2}TCE \leq 0$, $\frac{\partial^2}{\partial\alpha_2\partial\sigma_1^2}TCE = 0$

- Comp Statics: I
  - If $\sigma_1^2 \downarrow\ \alpha_1 \uparrow\ \alpha_2 \uparrow$
  - If technology is a choice, then reinforcing choices: $\sigma_1^2 \downarrow\ \alpha_1 \uparrow\ \alpha_2 \uparrow$
  - Suppose we observe different firms, and that (σ_1^2, σ_2^2) vary independently in population. Each firm t gets a draw of (σ_1^2, σ_2^2).
  - Since $(\alpha_1, \alpha_2) \downarrow \left(\sigma_1^2, \sigma_2^2\right)$, there will be a positive correlation between α_1 and α_2.
  - We expect to see some firms with low (α_1, α_2) and some with high $(\alpha_1, \alpha_2) \Rightarrow$ in general expect to see clusters.
    * Some firms, however, will appear "mis-matched" due to high σ_1^2 and low σ_2^2.

- Application: Franchising
  - Royalty rate: proportion of profits paid to company
  - Asset maintainance: mandatory advertising

- Monitoring technology: inspection for quality
- Empirical prediction: company owned on interstate, franchises in local stores

- Note: interpretations of α_1, α_2 can be quite general (i.e. government-owned or privately owned, etc.) See Hart, Schliefer, Vischny on prisons.

- Common Agency (Holmstrom and Milgrom unpublished paper; also see Dixit)
  - Two principals
  - Two efforts
  - Case 1: Principal i cares about x_i, cannot observe x_j.
    * "Overworked student"
    * Exert externalities on other principal, over-incentivize agent
  - Case 2: Principal i cares about x_i, can observe both outputs.
    * Low-powered incentives for politicians (Dixit)
    * Incentive schemes cancel each other out