MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Economics

14.129 - Contract Theory, Fall 1999

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- Non-linear Pricing (see Fundenberg-Tirole Ch. 7)
  - Monopolist produces a good at constant marginal cost c, sells $q \geq 0$ to consumers
  - Consumer:

$$u(q,\theta) - t$$

  - Assumptions:

$$\begin{aligned} u(0,\theta) &= 0 \;\; \forall\theta \\ u_1(q,\theta) &> 0 \;\; \forall\theta > 0 \\ u_{11}(q,\theta) &< 0 \;\; \forall\theta < 0 \\ u_{12} &\geq 0 \text{ (supermodularity, } \Longrightarrow \text{SCP)} \\ u_2(q,\theta) &> 0 \text{ if } \forall q > 0 \end{aligned}$$

  - Game
    - Seller (P) specifies $\{(q,t)\}$ menu
      - equivalently: feasible q, and $t(q)$ are specified
    - Buyer accepts or rejects (IR)
      Note: if $t(0) = 0$, no rejection
    - If accepts the buyer chooses (q,t) pair to max $(u(q,\theta) - t(q))$
    - Note: as in moral hazard problem, we will reduce a game to a constrained optimization problem.

- Benchmark case (full info.)
  - principal obersves θ when customer arrives
  - P offers $q^*(\theta)$ = social optimum and charges

$$t(q^*(\theta)) = u(q^*(\theta), \theta)$$

  - Solving for the first-best quantity:

$$\max_q \left[u(q, \theta) - c(q)\right]$$

    FOC:

$$c(q) = cq \Rightarrow u_1(q, \theta) = c$$

- Private info, two types
  - Setup
    - $\theta \in \{\theta_L, \theta_H\}$
    - p_L share of θ_L in population; $p_H = 1 - p_L$
    - WLOG can think about two bundles:

$$\{(q_L, t_L), (q_H, t_H)\}$$

    - When will correct type choose correct bundles? If the following are satisfied:

$$\begin{aligned}
(IR_L) \;\; u(q_L, \theta_L) - t_L &\geq 0 \\
(IR_H) \;\; u(q_H, t_H) - t_H &\geq 0 \\
(IC_L) \;\; u(q_L, \theta_L) - t_L &\geq u(q_H, \theta_L) - t_H \\
(IC_H) \;\; u(q_H, \theta_H) - t_H &\geq u(q_L, \theta_H) - t_L
\end{aligned}$$

    - Principal:

$$\max_{q_L, q_H, t_L, t_H} \left[p_L(t_L - cq_L) + p_H(t_H - cq_H)\right]$$
$$s.t. \; (IR_L), \; (IR_H), \; (IC_L), \; (IC_H)$$

- Step 1: Consequences of SCP and $u_2 > 0$
  - (IR_L) and $(IC_H) \Rightarrow (IR_H)$ (since $u_2 > 0$)
  - (IC_H) binds implies that (IC_L) holds
  - (IC_L) binds $\Rightarrow (IC_H)$ satisfied (similar).
- Step 2: Optimality
  - IR_L should bind, IR_H does not
    - Proof: Suppose

$$u(q_L, \theta_L) - t_L > 0$$

Then, improve profits with the following adjustment:

$$\begin{aligned} t_L &\rightarrow t_L + \varepsilon, \\ t_H &\rightarrow t_H + \varepsilon \end{aligned}$$

  - IC_H binds

· Proof: Suppose

$$u(q_H, \theta_H) - t_H > u(q_L, \theta_H) - t_L$$

Then, improve profits with the following adjustment:

$$t_H \to t_H + \varepsilon.$$

* So: IR_L and IC_H are binding, but no other constraints.
* IR_L binds:

$$t_L(q_L) = u(q_L, \theta_L)$$

* IR_H binds:

$$u(q_H, \theta_H) - t_H = u(q_L, \theta_H) - t_L$$

Or,

$$t_H = u(q_H, \theta_H) - [\underbrace{u(q_L, \theta_H) - u(q_L, \theta_L)}_{\text{rent to the H-type}}]$$

– Principal's problem (substituting in):

$$\max_{q_L, q_H} \ [p_L(u(q_L, \theta_L) - c\ q_L) + p_H(u(q_H, \theta_H) - (u(q_L, \theta_H) - u(q_L, \theta_L)) - c\ q_H)]$$

No distortion the top: $q_H = q_H^*$ (choose the social optimum)

$$\begin{aligned} q_H \ &: \ u_1(q_H^*, \theta_H) = c \\ q_L \ &: \ p_L\ u_1(q_L, \theta_L) - p_H[\underbrace{u_1(q_L, \theta_H) - u_1(q_L, \theta_L)}_{>0}] = c\ p_L \end{aligned}$$

- Results:
  - q_H is optimal (no distortion on the top)
  - $q_L < q^*(\theta_L)$

- Many-type Model
  - $(\theta_{0,}..., \theta_n)$
    Principal offers a schedule $\{(q_0, t_0), ..., (q_n, t_n)\}$
    $\theta_i > \theta_{i-1}$
    $IC: \ \theta_i$ chooses (q_i, t_i)
  - Revelation Principle:
    - It is the equivalent to offer $\{(q_i, t_i)\}$ which are IC, or to ask agents to report θ_i and then assign (q_i, t_i)

- * Define indirect utility:

$$U(\widehat{\theta} \mid \theta) = u(q(\widehat{\theta}), \theta) - t(\widehat{\theta})$$

  in discrete type model:

$$U(\theta_j \mid \theta_i) = u(q(\theta_j), \theta_i) - t(\theta_i)$$

- * Incentive compatibility constrants:

$$\begin{aligned} IC(ij) &: U(\theta_j \mid \theta_i) \leq U(\theta_j \mid \theta_i) \\ IR(i) &: U(\theta_i \mid \theta_i) \geq 0 \text{ (ind. rationality)} \end{aligned}$$

- Constraint Reduction Theorem:

  Assume SCP.

  $IC(ij)$ is satisfied $\forall i, j$ iff q is nondecreasing, and $DLIC$ (downward local incentive compatibility)

$$U(\theta_i \mid \theta_i) \geq U(\theta_{i-1} \mid \theta_i) \;\; (i = 1, .., n)$$

  and $ULIC$ (upward local incentive compatibility)

$$U(\theta_i \mid \theta_i) \geq U(\theta_{i+1} \mid \theta_i) \;\; (i = 1, .., n-1)$$

  are satisfied.

  – Note: There are n^2 constraints of the form $IC(i, j)$.
    Constraint reduction theorem: there are $2n$ constraints in $(DLIC)+$ $(ULIC)$

  – Note: $SCP + IC \Rightarrow q(\theta) \uparrow \theta$

- Proof:

$$DLIC(1) + DLIC(2) \Rightarrow IC(2, 0)$$

- SCP orders slopes of IC; $IC(i, i-1)$ gives positions of points on IC

$$ULIC(0) + ULIC(1) \Rightarrow IC(0, 2)$$

- The following results from two types generalized for N-types case
  - $IR(0)$ will bind
  - $IR(0) + DLIC(1) \Rightarrow IR(1)$
    $IR(i) + DLIC(i+1) \Rightarrow IR(i+1)$
  - $DLIC$ will bind
    $ULIC$ will be slack
  - Any non-decreasing $q(\theta)$ can be implemented
  - q_n is efficient
    q_i for $i < n$ is inefficiently low

- Continuous-Type Model
  - $F(\theta)$ is a c.d.f. $\theta \in [\theta, \overline{\theta}]$
  - $f(\theta)$ is a corresponding p.d.f
  - $\forall\, \theta \in [\theta, \overline{\theta}]$, $f(\theta) > 0$
  - Preferences of agent:

$$u(q,t) - t$$

  - Preferences of principal:

$$v(q,\theta) + t$$

  - SCP:

$$\begin{aligned} \frac{\partial^2 u}{\partial q \partial \theta} &\geq 0 \\ \frac{\partial u}{\partial \theta} &> 0 \end{aligned}$$

  - IC:

$$U(\theta \mid \theta) \geq U(\widehat{\theta} \mid \theta) \quad \forall \widehat{\theta}$$

- Constraint Reduction Theorem

  SCP is satisfied

  - (i) $\{q(\theta), t(\theta)\}$ satisfies IC
  - IF AND ONLY IF
  - (ii) (a) $q(\theta)$ is non-decreasing

– (ii) (b)

$$\underbrace{\frac{d}{d\theta}U(\theta \mid \theta) = U_2(\theta \mid \theta) = u_2(q(\theta), \theta)}_{\text{Local } IC}$$

almost everywhere. (in particular, at all points whereU_2 exists, which is a.e.)

- Proof

  – (i) ⇒ (ii)

  – (ii)(a) holds by a SCP, indeed:

$$\begin{aligned} (IC) \;\; &\Leftrightarrow \;\; \theta = \arg\max_{\widehat{\theta}} U(\widehat{\theta} \mid \theta) \Leftrightarrow \\ q(\theta) \;\; &= \;\; \arg\max_{q(\widehat{\theta})} [u(q(\widehat{\theta}), \theta) - t(q(\widehat{\theta}))] \end{aligned}$$

by SCP, $q(\widehat{\theta})$ is non-decreasing

  – (ii)(b) since $\widehat{\theta}$ is drawn from a compact set and $U(\widehat{\theta} \mid \theta)$ is differentiable in θ, apply envelope theorem, since $(IC) \Rightarrow$

$$U(\theta \mid \theta) = \max_{\widehat{\theta}} U(\widehat{\theta} \mid \theta).$$

  – (ii)⇒ (i)
  Suppose IC fails, then

$$\exists\, \theta \neq \widehat{\theta}, \text{ s.t. } U(\widehat{\theta} \mid \theta) - U(\theta \mid \theta) > 0 \Leftrightarrow \int_{\theta}^{\widehat{\theta}} U_1(s \mid \theta)\, ds > 0$$

since $U_1(s \mid s) = 0$ a.e.,

$$\int_{\theta}^{\widehat{\theta}} [U_1(s \mid \theta) - U_1(s \mid s)]\, ds > 0$$

$$-\int_{\theta}^{\widehat{\theta}} \int_{\theta}^{s} U_{12}(s \mid v)\, dv\, ds > 0$$

$$U_{12}(s \mid v) = \frac{\partial^2}{\partial s \partial v}[u(q(s), v) - t(q(s))] = u_{12}(q(s), 0)q'(s) \geq 0$$

But, $q' \geq 0$ (q diff. a.e., since it is monotone) and the SCP imply that

$$-\int_{\theta}^{\widehat{\theta}} \int_{\theta}^{s} \underbrace{u_{12}\, q'}_{\geq 0} dv\, ds > 0,$$

a contradiction. Thus, IC holds.

– Now, observe that by the fundamental theorem of calculus,

$$\begin{aligned} U(\theta \mid \theta) &= U(\underline{\theta} \mid \underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{d}{ds} U(s \mid s)\, ds \\ &= U(\underline{\theta} \mid \underline{\theta}) + \int_{\underline{\theta}}^{\theta} U_2(s \mid s)\, ds \\ &= U(\underline{\theta} \mid \underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_2(q(s), s)\, ds \end{aligned}$$

– Suppose $u = q \cdot \theta \Rightarrow u_2 = q$

$$U(\theta \mid \theta) = U(\underline{\theta} \mid \underline{\theta}) + \underbrace{\int_{\underline{\theta}}^{\theta} q(s)\, ds}_{\text{efficiency rent}}$$

– Efficiency rent: there are externalities among types. Giving more quantity to low types increases surplus to high types.

- Principal's problem:

$$\max_{q(\theta), t(\theta)} E\left[v(q(\theta), \theta) + t(q(\theta))\right]$$

– Social surplus:

$$s(q(\theta), \theta) = v(q(\theta), \theta) + u(q(\theta), \theta)$$

– Prinicipal's objective function is:

$$s(q(\theta), \theta) - U(\theta \mid \theta) \Rightarrow \max_{q(\theta), t(\theta)} E\left\{s(q(\theta), \theta) - U(\theta \mid \theta)\right\} \text{ s.t. } IR,\ IC$$

– Use constraint reduction theorem and fact $IC + IR(\underline{\theta}) \Rightarrow IR(\theta)$, that is $U(\underline{\theta} \mid \underline{\theta}) \geq 0$

$$IC : U(\theta \mid \theta) = U(\underline{\theta} \mid \underline{\theta}) + \int_{\underline{\theta}}^{\theta} U_2(s \mid s)\, ds$$

Relax constraint that q is non-decreasing and check later.

$$\max \int_{\underline{\theta}}^{\overline{\theta}} [s(q(\theta), \theta) - \int_{\underline{\theta}}^{\theta} u_2(q(s), s)\, ds] f(\theta) d\theta$$

– Then, we can write:

$$
\begin{aligned}
E[U(\theta \mid \theta)] &= \int_{\underline{\theta}}^{\overline{\theta}} U(\theta \mid \theta)\, f(\theta) d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\theta} u_2(q(s), s)\, ds \right) f(\theta) d\theta \\
&= -U(\theta \mid \theta)(1 - F(\theta))|_{\underline{\theta}}^{\overline{\theta}} + \int_{\underline{\theta}}^{\overline{\theta}} (1 - F(\theta)) u_2(q(\theta), \theta) d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} u_2(q(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} f(\theta) d\theta = E[u_2(q(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)}]
\end{aligned}
$$