Bond percolation does not simulate site percolation

Nikita Gladkov*[1] and Aleksandr Zimin†[2]

[1]Department of Mathematics, UCLA, Rényi institute
[2]Department of Mathematics, MIT

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Abstract

We show that a site percolation is a stronger model than a bond percolation. We use the van den Berg – Kesten (vdBK) inequality to prove that site percolation on a neighborhood of a vertex of degree 3 cannot be simulated even approximately by bond percolation, and develop a decision tree technique to prove the same for a neighborhood of a vertex of degree 2. This technique can be used to obtain inequalities for connectedness probabilities, including a conjectured inequality of Erik Aas.

1 Introduction

Assume that we have a graph $G = (V, E)$ and run a Bernoulli bond percolation on it with every *edge* $e \in E$ having its own probability p_e of being open independently on other edges. A Bernoulli site percolation, on the contrary, is a process in which every *vertex* $v \in V$ has a probability of being open. One can ask many questions about the probabilistic properties of clusters connected via open vertices and edges. There are known inequalities for the critical probabilities of site and bond percolation on the same infinite graph [GS98].

To motivate our problem, recall Exercise 3.4 in [G18] (see also Exercise 6 in [DC18]): "Show that bond percolation on a graph G may be reformulated in terms of site percolation on a graph derived suitably from G." Here is a formal definition.

Definition 1.1. One says that vertices v and u are connected (belong to the same cluster) in a site/bond percolation if there is a path between them passing only through open sites/bonds. In the case of site percolation, we also require that v and u be open themselves.

We say that a site/bond percolation μ_1 on graph $G' = (V', E')$ *simulates* a bond/site percolation μ_2 on graph $G = (V, E)$ if there is a function $f : V \to V'$ such that for all events "v is connected to u" and their boolean combinations, their probability in μ_1 is equal to the probabilities of the events "$f(v)$ is connected to $f(u)$" in μ_2 and their corresponding boolean combinations.

Remark 1.2. By this definition, the simulation preserves events such as "At least n out of m vertices $v_1, \ldots, v_m$ are in the same cluster", but is not guaranteed to preserve the probability of "There is a path from a to b avoiding vertex c".

Then the solution to the exercise is given by the following theorem [F61, FE61].

Theorem 1.3. *For every graph G and a bond percolation μ_b on it there exists a graph G' with a site percolation μ_s which simulates μ_b.*

*gladkovna@ucla.edu
†azimin@mit.edu

Proof. Let G' be a copy of G with an additional auxiliary vertex in the middle of each edge. Make all original vertices open with probability 1 and every auxiliary vertex in the middle of each edge e open with probability p_e. This site percolation on G' is precisely isomorphic to the bond percolation on G. □

Similarly, it is natural to ask whether site percolation can be simulated by bond percolation. Fisher [F61] noted that the other direction can not be true since the argument proving Theorem 1.3 is only invertible for line graphs. We make his argument precise in Theorem 2.4 proved in Section 2. However, the question becomes more interesting if we consider approximate simulations.

Definition 1.4. We say that a sequence of site (bond) percolations $\{\mu_i\}$ on graphs $G_i = (V_i, E_i)$ *approximately simulates* a bond (site) percolation ν on graph $G = (V, E)$ if there are functions $f_i : V \to V_i$ such that probabilities of the events "$f_i(v)$ is connected to $f_i(u)$" in μ_i and all their Boolean combinations tend to the probabilities of the corresponding Boolean combinations of the events "v is connected to u" in ν.

The main results of the paper are the following theorems:

Theorem 1.5. *One cannot approximately simulate site percolation on the complete bipartite graph $K_{1,3}$ (claw graph) by bond percolation.*

This is the corollary of Theorem 3.2.

Theorem 1.6. *One cannot approximately simulate a site percolation on the path of length 2 by bond percolation.*

Similarly, it is the corollary of Theorem 4.6. This proof requires new inequalities concerning connectedness events in percolation. We use computer search techniques to discover new inequalities, including inequalities (5) and (6) in Section 5.

2 Preliminary remarks

Definition 2.1. A full hyperedge Bernoulli percolation is a random model on a hypergraph $H = (V, E)$. Every hyperedge e has a probability p_e of being open. The vertices v and u are said to be connected if there is a path between them such that each edge in the path is a subset of some open hyperedge.

Let us show that full hyperedge percolation is equal in power to site percolation.

Theorem 2.2. *Full hyperedge percolation simulates site percolation and vice versa.*

Proof. To simulate full hyperedge percolation by site percolation, for every hyperedge with probability p_e we add one additional vertex with probability p_e and connect it to all elements of the hyperedge. All the original vertices stay open with probability 1.

Conversely, to simulate the site percolation by full hyperedge percolation, we first consider a graph G' from the proof of Theorem 1.3. By the construction, vertices of G' can be original or auxiliary. For each original vertex v we add a hyperedge e_v with probability p_v, connecting v and all adjacent auxiliary vertices. It is easy to see that all connectivity events are preserved by these simulations. □

Note that the hypergraph percolation in the sense of [WZ11] is more general than our full hypergraph percolation and is capable of modeling more phenomena. We may use our reduction

So, simulating site percolation is equivalent to simulating full hyperedge percolation. It is easy to see that bond percolation cannot simulate *exactly* even a hyperedge of size 3 with probability $0 < p < 1$, thus proving Fisher's remark. Indeed, a model of such a hyperedge would be some graph G with bond percolation on it.

Definition 2.3. We denote by $v_{11}v_{12}\dots v_{1i_1}|v_{21}\dots v_{2i_2}|v_{n1}\dots v_{ni_n}$ the event that the vertices $v_{11}, \dots, v_{1i_1}$ belong to the same cluster, vertices $v_{21}, \dots, v_{2i_2}$ belong to the same cluster, ..., vertices $v_{n1}, \dots, v_{ni_n}$ belong to the same cluster, and, moreover, these clusters are different. By $\mathbf{P}(v_{11}v_{12}\dots v_{1i_1}|v_{21}\dots v_{2i_2}|v_{n1}\dots v_{ni_n})$ we denote the probability of this event in the underlying bond percolation. In particular, $\mathbf{P}(abc)$ denotes the probability that a, b and c are in the same cluster and $\mathbf{P}(a|b|c)$ is the probability that a, b and c are in 3 different clusters.

Theorem 2.4. *For all $0 < p < 1$, every simple graph $G = (V, E)$ and vertices $a, b, c \in V$ one has either $\mathbf{P}(abc) < p$ or $\mathbf{P}(a|b|c) < 1 - p$, where $\mathbf{P}$ is taken over random subgraphs given by a bond percolation on G.*

Proof. Remove all edges in G that have a probability of 0 and contract all edges with a probability of 1. Now, none of the edges in G is certain.

First, assume that there is a path P from a to b not passing through c. Then there is a nonzero probability that all edges of P will be open and the remaining edges will be closed, so $\mathbf{P}(ab|c) > 0$, where $\mathbf{P}(ab|c)$ is the probability that a and b are in the same component and c in the other. This contradicts the equation $\mathbf{P}(abc) + \mathbf{P}(a|b|c) = 1$, so every path from a to b should pass through c.

Similarly, every path from a to c should pass through b, but that means that there are no paths from a to b or c, since any such path will first go to b or c. Thus $p = 0$, which contradicts the assumptions. □

By this theorem, it is impossible to simulate the hyperedge percolation by bond percolation, we consider a question if it is possible to have an arbitrary good approximation.

Question 2.5. *For given k, $p \notin \{0, 1\}$ and $\varepsilon > 0$, does there exist a graph $G = (V, E)$, containing vertices x_1, $\ldots$, x_k and a bond percolation on it with $\mathbf{P}(x_1 x_2 \ldots x_k) > p - \varepsilon$ and $\mathbf{P}(x_1|x_2|\ldots|x_k) > 1 - p - \varepsilon$?*

In Section 3 we show that approximate simulation is impossible for $k \geq 4$ using a lemma due to Hutchcroft [H21], thus proving Theorem 1.5. Finally, we develop a new technique using decision trees to resolve Question 2.5 for $k = 3$ (thereby proving Theorem 1.6) in Section 4.

3 Simulating k-hyperedge for $k \geq 4$

In [H21], the following theorem is proved using the vdBK inequality, where K_u is the cluster containing vertex u, and for each finite subset $\Lambda \subseteq V$

$$|K_{max}(\Lambda)| = \max\{|K_v \cap \Lambda| : v \in V\}$$

is the maximal number of vertices from Λ belonging to the same cluster.

Theorem 3.1 ([H21], Theorem 2.3)**.** *Let $G = (V, E)$ be a countable graph and let $\Lambda \subseteq V$ be finite and non-empty. Then for Bernoulli bond percolation one has*

$$\mathbf{P}(|K_{max}(\Lambda)| \geq 3^k \lambda) \leq \mathbf{P}(|K_{max}(\Lambda)| \geq \lambda)^{3^{k-1}+1} \tag{1}$$

and

$$\mathbf{P}(|K_u \cap \Lambda| \geq 3^k \lambda) \leq \mathbf{P}(|K_{max}(\Lambda)| \geq \lambda)^{3^{k-1}} \mathbf{P}(|K_u \cap \Lambda| \geq \lambda) \tag{2}$$

for every $\lambda \geq 1$ (not necessarily integer), integer $k \geq 0$ and $u \in V$.

This allows us to prove that one cannot even approximately simulate the 4-hyperedge.

Theorem 3.2. *For all $0 < p < 1$ there exists an $\varepsilon > 0$ such that for any graph $G = (V, E)$, bond percolation on it and vertices $a, b, c, d \in V$ one has either $\mathbf{P}(abcd) < p - \varepsilon$ or $\mathbf{P}(a|b|c|d) < 1 - p - \varepsilon$.*

Proof. Assume that such a graph G exists. Let Λ be $\{a, b, c, d\}$. Then (1) with $\lambda = \frac{4}{3}$ gives

$$\mathbf{P}(abcd) \leq \mathbf{P}(ab \cup ac \cup ad \cup bc \cup bd \cup cd)^2.$$

If the statement of the theorem were false, this would imply

$$p - \varepsilon \leq (p + \varepsilon)^2,$$

which is false for small ε. □

Theorem 3.2 implies Theorem 1.5. Indeed, the 4-hyperedge can be simulated by site percolation on a claw graph, but not by any bond percolation.

4 Simulating 3-hyperedge: human proof

Now we see that it is impossible to even approximately simulate site percolation with bond percolation for the claw graph, as promised in Theorem 1.5. To prove Theorem 1.6, we need the following lemma.

Definition 4.1. For two configurations $C_1, C_2 \in \Omega = 2^{[E]}$ and a set $S \subseteq E$ we denote by $C_1 \to_S C_2$ the configuration which coincides with C_1 on S and C_2 on its complement $\overline{S}$.

Lemma 4.2. *Consider two independent Bernoulli bond percolations C_1 and C_2 having the same distribution μ on the same graph G. Let a decision tree T select each edge and reveal it in both C_1 and C_2. Furthermore, allow on each step, before revealing, to decide if this edge will go to the set S (thus dependent on C_1 and C_2) or to its complement $\overline{S}$. Then $C_1 \to_S C_2$ is independent of $C_2 \to_S C_1 = C_1 \to_{\bar{S}} C_2$ and both of them are distributed as μ.*

Example 4.3. If the graph is a path of length 2 from a to b, then the tree T on Figure 1 builds a set S of all edges with one end in the component of a in C_1.

Proof of Lemma 4.2. For finite graphs with $|E| = n$ and for every pair of configurations C_3, C_4 there is only one path in any decision tree leading to $C_1 \to_{S(C_1,C_2)} C_2 = S_3$ and $C_1 \to_{\bar{S}(C_1,C_2)} C_2 = S_4$ and the probability of this path is equal to $\mathbf{P}(C_1)\mathbf{P}(C_2)$, which is equal to $\mathbf{P}(C_3)\mathbf{P}(C_4)$ since the probability in Bernoulli percolation is a product of probabilities for individual edges. □

Example 4.4. For example, one can take T querying all the edges from the vertices already known to connect to the vertex a in C_1. It will assign all these edges to S and then discover the remaining edges assigning them to $\bar{S}$. Then S will be the set of all open and closed edges with at least one edge in the component of a. Lemma 4.2 claims that rerunning the choice for these edges will result in measure μ and rerunning the choice for the remaining edges will also result in measure μ.

Remark 4.5. Notice that Markov chain method from [BHK06] is based on the fact that rerunning the choice for edges in $\overline{S}$ from Example 4.4 preserves the measure restriction $\mu|_{a|b}$.

In our notation, it means that for $A = a|b$ and any B one has

$$\mathbf{P}(C_1 \in A \text{ and } C_1 \underset{S(C_1)}{\to} C_2 \in B) = \mathbf{P}(A \cap B) \tag{3}$$

Theorem 4.6. *For all $0 < p < 1$ there exists an $\varepsilon > 0$ such that for any graph $G = (V, E)$, bond percolation on it and vertices $a, b, c \in V$ one has either $\mathbf{P}(abc) < p - \varepsilon$ or $\mathbf{P}(a|b|c) < 1 - p - \varepsilon$.*

Proof. We will need multiple sets S_i for our purpose. So, we define sets S_1, S_2 and S_3, which are somewhat complex (See Figure 2).

To build S_1, we query all edges connected to c and put them in S. Then we query all not queried edges connected to a (this is vacuous if a was connected to c) and put them in $\bar{S}$. Then we query all not queried edges connected to b and put them in S. Finally, we put the rest of the edges in $\bar{S}$. If we denote by Com_x the set of vertices connected to x via edges open in C_1, we get

$$S_1 = \begin{cases} E \cap \left(Com_c \times V \cup Com_b \times \overline{Com_a}\right) \text{ if } C_1 \in a|b|c; \\ E \cap \left(Com_c \times V\right) \text{ if } C_1 \text{ is in } abc,\, a|bc \text{ or } ab|c; \\ E \cap \left((Com_b \cup Com_c) \times V\right) \text{ if } C_1 \in ac|b. \end{cases}$$

The only case we will actually use is $a|b|c$. S_2 is defined analogously with b and c interchanged.

$$S_2 = \begin{cases} E \cap \left(Com_b \times V \cup Com_c \times \overline{Com_a}\right) \text{ if } C_1 \in a|b|c; \\ E \cap \left(Com_b \times V\right) \text{ if } C_1 \text{ is in } abc,\, a|bc \text{ or } ac|b; \\ E \cap \left((Com_b \cup Com_c) \times V\right) \text{ if } C_1 \in ab|c. \end{cases}$$

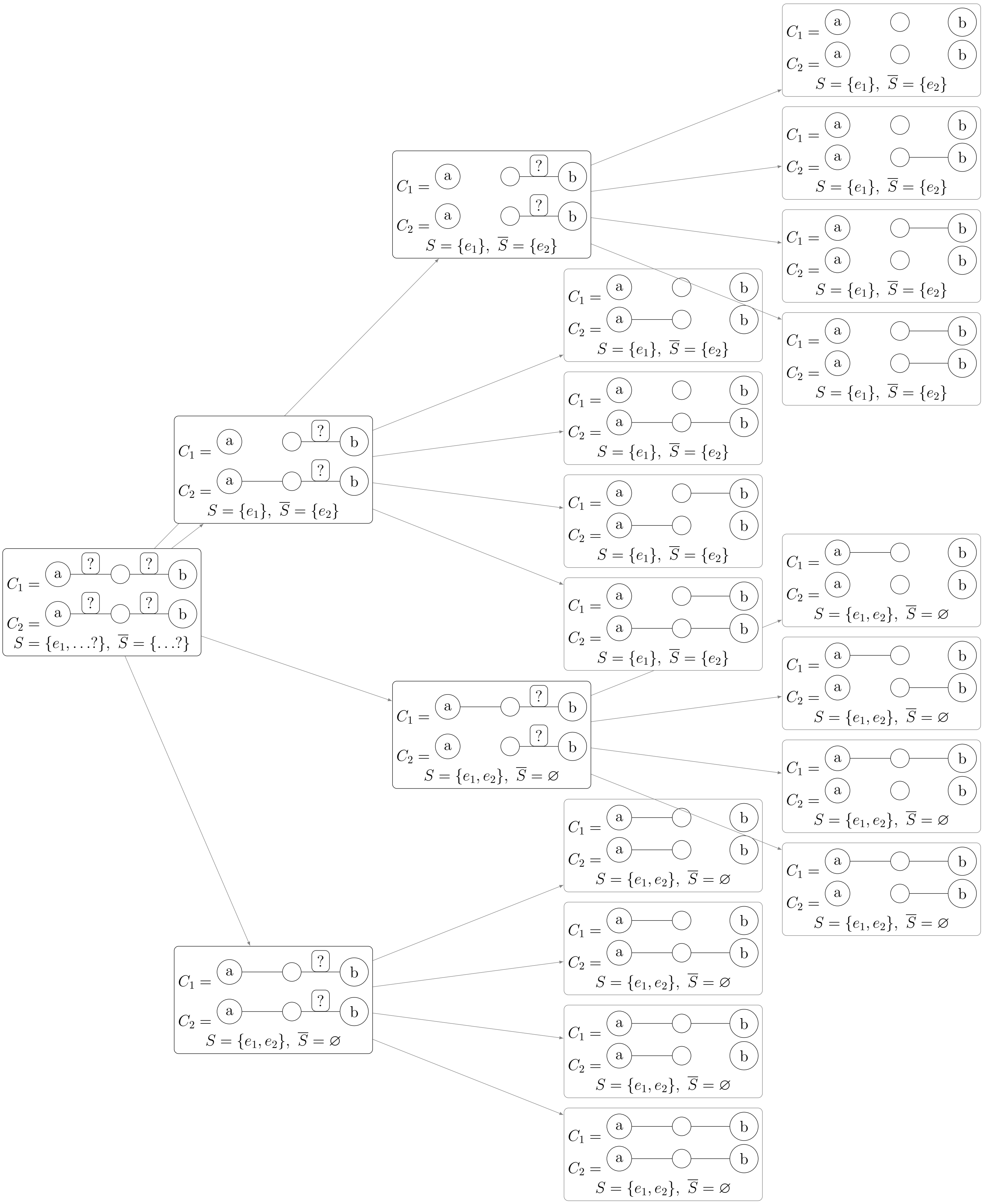

Figure 1: T corresponding to the Example 4.3

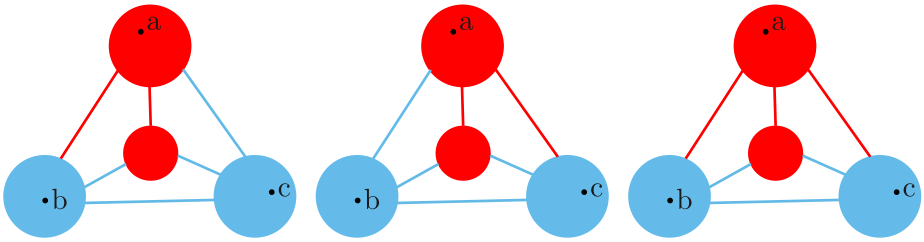

Figure 2: S_1, S_2 and S_3 for the case $C_1 \in a|b|c$. Regions surrounding a, b, c depict Com_a, Com_b and Com_c. Respective sets are in blue and their complements are in red.

Finally, for S_3 we put all edges connected to a in $\bar{S}$, all not queried edges connected to b or c to S and the rest of the edges to $\bar{S}$.

$$S_3 = \begin{cases} E \cap \left((Com_b \cup Com_c) \times \overline{Com_a}\right) \text{ if } C_1 \in a|b|c; \\ \varnothing \text{ if } C_1 \in abc; \\ \text{Something else otherwise.} \end{cases}$$

The key observation is that when $C_1 \in a|b|c$ and $C_1 \to_{S_3} C_2 \in ab \cup ac$, one has $C_1 \to_{S_1} C_2 \in ab$ or $C_1 \to_{S_2} C_2 \in ac$. Indeed, there is a path p from a to b or c in $C_1 \to_{S_3} C_2$. There should be the first edge e where p goes to $Com_b \cup Com_c$. The path segment before e is contained in all of $\bar{S_1}$, $\bar{S_2}$ and $\bar{S_3}$. The edge e itself belongs to $Com_a \times (Com_b \cup Com_c)$ and so to $\bar{S_1}$ or $\bar{S_2}$. Since all internal edges in Com_b and Com_c belong to S_1 and S_3, we get $C_1 \to_{S_1} C_2 \in ab$ or $C_1 \to_{S_2} C_2 \in ac$.

Now, let's proceed to estimate the probabilities of these events. For $C_1 \in a|b|c$ we will have $C_1 \to_{S_1} C_2 \in a|c$, so

$$\mathbf{P}(C_1 \in a|b|c \text{ and } C_1 \underset{S_1}{\to} C_2 \in ab) \leq \mathbf{P}(ab|c).$$

Similarly,

$$\mathbf{P}(C_1 \in a|b|c \text{ and } C_1 \underset{S_2}{\to} C_2 \in ac) \leq \mathbf{P}(ac|b).$$

To estimate the probability of $\mathbf{P}\left(C_1 \in a|b|c \text{ and } C_1 \to_{S_3} C_2 \in (ab \cup ac)\right)$ we make use of the fact that if C_1 belongs to $a|b \cap a|c$, then $\bar{S_3}$ contains a cut from a to b and c, so $C_1 \to_{\bar{S_3}} C_2$ also belongs to $a|b \cap a|c$.

$$\begin{aligned} &\mathbf{P}\Big(C_1 \in a|b|c \text{ and } C_1 \underset{S_3}{\to} C_2 \in (ab \cup ac)\Big) \\ &\qquad \geq \mathbf{P}\Big(C_1 \in (a|b \cap a|c) \text{ and } C_1 \underset{S_3}{\to} C_2 \in (ab \cup ac)\Big) - \mathbf{P}(a|bc) \\ &= \mathbf{P}\Big(C_1 \underset{\bar{S_3}}{\to} C_2 \in (a|b \cap a|c) \text{ and } C_1 \underset{S_3}{\to} C_2 \in (ab \cup ac)\Big) - \mathbf{P}(a|bc) = \mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) - \mathbf{P}(a|bc) \end{aligned}$$

Finally, this allows us to make the conclusion

$$\mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) \leq \mathbf{P}(ab|c) + \mathbf{P}(ac|b) + \mathbf{P}(a|bc). \tag{4}$$

If $\mathbf{P}(abc) \geq p - \varepsilon$ and $\mathbf{P}(a|b|c) \geq 1 - p - \varepsilon$, this implies $(p-\varepsilon)(1-p-\varepsilon) \leq 2\varepsilon$, which is false for small ε. □

Theorem 1.6 follows from here.

For $p = \frac{1}{2}$, from the equation (4) one can conclude that $\mathbf{P}(abc)$ and $\mathbf{P}(a|b|c)$ can not be simultaneously greater than 0.37586. If we denote the maximal possible value of $\min\big(\mathbf{P}(abc), \mathbf{P}(a|b|c)\big)$ for any bond percolation by α_3, we get an estimate $\alpha_3 < 0.37586$, which we improve in the next section. The lower bound $\alpha_3 > 0.29065$ is given in Appendix A.

5 Simulating 3-hyperedge: computer-assisted proof

Assume that we have a graph G with designated vertices a, b, c and a bond percolation on it. Let S_1, S_2 and S_3 be as in the previous section. Bond percolation induces a distribution ρ on $J = \{a|b|c, a|bc, ac|b, ab|c, abc\}$. In the same way, consider all possible 8-tuples

$$(C_1, C_2, C_1 \underset{S_1}{\to} C_2, C_1 \underset{\overline{S_1}}{\to} C_2, C_1 \underset{S_2}{\to} C_2, C_1 \underset{\overline{S_2}}{\to} C_2, C_1 \underset{S_3}{\to} C_2, C_1 \underset{\overline{S_3}}{\to} C_2)$$

and the probability distribution they induce on J^8. Some of the elements of J^8 are impossible for graph restrictions. We find these impossible elements by an algorithm[1].

Also, by Lemma 4.2, this probability distribution should have the same marginals $\rho \times \rho$ when restricted to pairs (C_1, C_2), $(C_1 \to_{S_1} C_2, C_1 \to_{\overline{S_1}} C_2)$, $(C_1 \to_{S_2} C_2, C_1 \to_{\overline{S_2}} C_2)$ and $(C_1 \to_{S_3} C_2, C_1 \to_{\overline{S_3}} C_2)$. When we fix ρ, these restrictions produce a linear program, and one can see if it is feasible for different ρ's. One of the emerging restrictions on ρ is

$$\mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) \leq \mathbf{P}(ab|c) + \mathbf{P}(ac|b) + \mathbf{P}(a|bc) - \mathbf{P}(ab|c)^2 - \mathbf{P}(ac|b)^2, \tag{5}$$

which is obviously better then inequality (4) and leads to an estimate $\alpha_3 \leq 0.369$. We double-checked manually that the tuples $(J_1, J_2, \ldots, J_8) \in J^8$ where dual potentials of the linear program add up to the negative number are indeed infeasible.

Moreover, surprisingly, it also proves inequality

$$\mathbf{P}(abc)\mathbf{P}(a|b|c) \geq \mathbf{P}(ab|c)\mathbf{P}(ac|b) + \mathbf{P}(ab|c)\mathbf{P}(a|bc) + \mathbf{P}(ac|b)\mathbf{P}(a|bc), \tag{6}$$

which was first conjectured in an unpublished work of Erik Aas and proved in [G24]. It is stronger than what Harris–Kleitman inequality can tell about these events. Both of these computer-assisted proofs are available in the GitHub repository.

6 Further questions

Inequalitites (4) and (5) prove that if all three probabilities $\mathbf{P}(ab|c)$, $\mathbf{P}(ac|b)$ and $\mathbf{P}(a|bc)$ are 0, then one of $\mathbf{P}(abc)$ and $\mathbf{P}(a|b|c)$ should be 0. In fact, the stronger statement holds:

Proposition 6.1. *If* $\mathbf{P}(ab|c) = 0$, *then*

$$\mathbf{P}(a|b|c)\mathbf{P}(abc) = \mathbf{P}(ac|b)\mathbf{P}(a|bc) \text{ and } \mathbf{P}(abc) = \mathbf{P}(ac)\mathbf{P}(bc).$$

Proof. As in the proof of Theorem 2.4, we first delete or contract certain edges. Now all paths from a to b should pass through c, otherwise, there will be a nonzero probability of one such path being open and the rest of the edges closed. This means c splits the graph in halves, events ac and bc are determined by different sets of edges.

□

However, contrary to the inequalitites (4) and (5), this proof tells nothing when $\mathbf{P}(ab|c) < \varepsilon$. So, we pose two conjectures increasing in strength:

Conjecture 6.2. *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\mathbf{P}(ab|c) < \delta$ *and* $\mathbf{P}(ac|b) < \delta$, *then* $\mathbf{P}(abc)$ *or* $\mathbf{P}(a|b|c)$ *is less then* ε.

[1] https://github.com/Kroneckera/bunkbed
We loop through each element $(J_1, J_2, \ldots, J_8)$ of J^8. First, we split the vertices of G into "codes" based on which of the vertices a, b, c it lies together with in each of the configurations $(C_1, C_2, C_1 \to_{S_1} C_2, C_1 \to_{\overline{S_1}} C_2, C_1 \to_{S_2} C_2, C_1 \to_{\overline{S_2}} C_2, C_1 \to_{S_3} C_2, C_1 \to_{\overline{S_3}} C_2)$. Then we build "universal" graphs $\bar{G}_1$ and $\bar{G}_2$, using these codes as vertices, including all edges except for the edges between codes in different parts of the graph. Finally, we use $\bar{G}_1$ and $\bar{G}_2$ as C_1 and C_2, construct the remaining elements of the 8-tuple and use graph search algorithms to check whether they coincide with $(J_1, J_2, \ldots, J_8)$.

Conjecture 6.3. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{P}(ab|c) < \delta$, then*

$$\mathbf{P}(abc) - \mathbf{P}(ac)\mathbf{P}(bc) < \varepsilon.$$

Remark 6.4. On the contrary, if $\mathbf{P}(abc) - \mathbf{P}(ac)\mathbf{P}(bc) < \varepsilon$ then, by inequality (6),

$$\mathbf{P}(ab|c) < \frac{\varepsilon}{\mathbf{P}(ac \text{ or } bc)}.$$

It would also be interesting to find the exact value for α_3. The best boundaries are given in the Appendix A.

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References

[BK85] Jacob van den Berg and Harry Kesten. Inequalities with applications to percolation and reliability. *J. Appl. Probab.* **22.3** (1985): 556-569.

[BHK06] Jacob van den Berg, Olle Häggström, and Jeff Kahn. Some conditional correlation inequalities for percolation and related processes. *Random Structures & Algorithms* **29.4** (2006): 417-435.

[DC18] Hugo Duminil–Copin. Introduction to Bernoulli percolation. *Lecture notes available on the webpage of the author* (2018), 5 pp.

[F61] Michael E. Fisher. Critical probabilities for cluster size and percolation problems. *J. Mathematical Phys.* **2.4** (1961): 620-627.

[FE61] Michael E. Fisher and John W. Essam. Some cluster size and percolation problems. *J. Mathematical Phys.* **2.4** (1961): 609-619.

[G24] Nikita Gladkov. A strong FKG inequality for multiple events, To appear in Bull. Lond. Math. Soc, (2024), 7 pp.

[GP24] Nikita Gladkov and Igor Pak. Positive dependence for colored percolation, *Phys. Rev. E* **109** (2024), 9 pp.

[G18] Geoffrey Grimmett. *Probability on graphs: random processes on graphs and lattices.* Vol. **8**. Cambridge University Press, (2018).

[GS98] Geoffrey Grimmett and Alan Stacey. Critical probabilities for site and bond percolation models, *Ann. Probab.* **26 (4)** (1998), 30 pp.

[H21] Tom Hutchcroft. Power-law bounds for critical long-range percolation below the upper-critical dimension. *Probab. Theory Related Fields* **181** (2021): 533-570.

[W21] Adam Zsolt Wagner. Constructions in combinatorics via neural networks, arXiv:2104.14516, 2021, 23 pp.

[WZ11] John C. Wierman and Robert M. Ziff. Self-dual planar hypergraphs and exact bond percolation thresholds. *Electron. J. Combin.* **18.1** (2011), 19 pp.

A Appendix: optimizing α_3

Let us recall that α_3 denotes the largest possible value of $\min\big(\mathbf{P}(abc), \mathbf{P}(a|b|c)\big)$ for the bond percolation. Let us restrict ourselves to the triangle graph with all three probabilities equal to p. Then $\mathbf{P}(a|b|c) = (1-p)^3$ and $\mathbf{P}(abc) = p^3 + 3p^2(1-p)$. These numbers coincide for $p \approx 0.3473$, and we get $\alpha_3 \geq \mathbf{P}(abc) = \mathbf{P}(a|b|c) \approx 0.278$, a root of the equation $x^3 - 24x^2 + 3x + 1 = 0$.

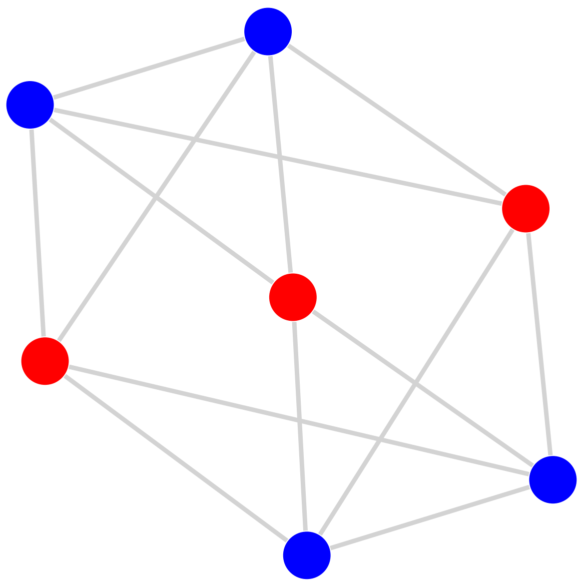

Figure 3: Graph for α_3.

One can do better by utilizing the graph in Figure 3 where each red-blue edge has probability 0.32537 and both blue-blue edges have probability 0.19231. This way we get $\mathbf{P}(abc) \approx \mathbf{P}(a|b|c) \approx 0.29065$.

Our computer search using algorithms from Wagner [W21] wasn't able to beat this estimate (See the best $\min\big(\mathbf{P}(abc), \mathbf{P}(a|b|c)\big)$ achieved on each training epoch in Figure 4).

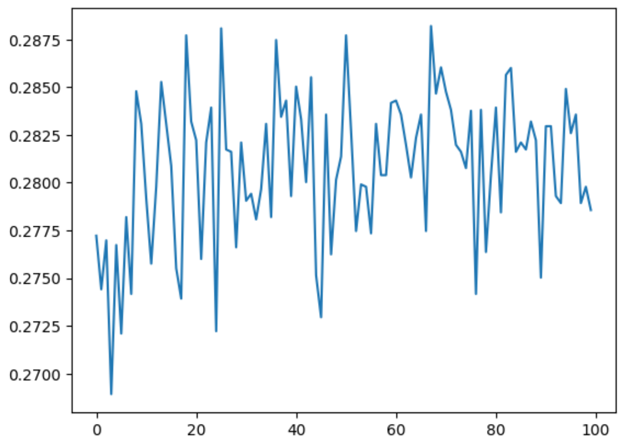

Figure 4: Best $\min\big(\mathbf{P}(abc), \mathbf{P}(a|b|c)\big)$ achieved on each training epoch.

In fact, if $\mathbf{P}(abc) = \mathbf{P}(a|b|c)$, it seems this probability can only lie in a narrow range from 0.27 to 0.291. Indeed, in this case inequality (6) gives the lower bound of $2 - \sqrt{3} \approx 0.2679$.