Inequalities in Graph Percolation

Based on the joint work with Nikita Gladkov

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Consider a graph $G = (V, E)$, where $V = \{1, 2, \dots, m\}$. Percolation is a random graph obtained from the graph G, where each edge $e \in E$ is independently open (or survives) with probability $p_e \in (0, 1)$. This gives a spanning subgraph $H \subseteq G$ with probability

$$\prod_{e \in H} p_e \prod_{e \notin H} (1 - p_e).$$

Percolation on infinite graphs is defined in the same manner.
A *cluster* is a set of vertices connected via open edges.

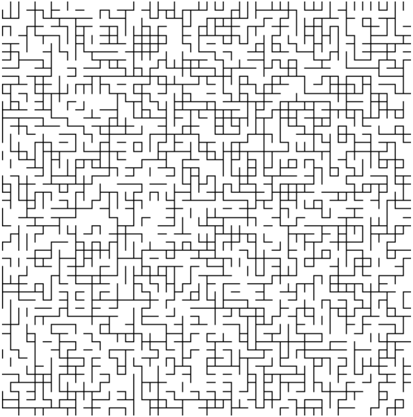
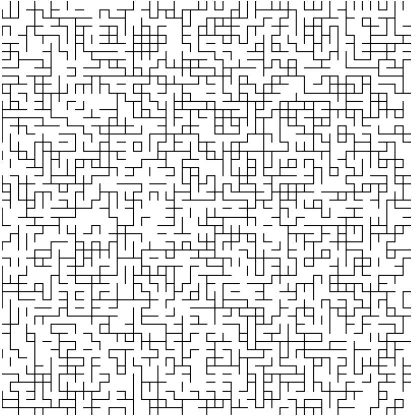


Figure: Percolation on $\mathbb{Z}^2$ with $p = 0.51$. Since $\mathbb{Z}^2$ is dual to itself, this picture can also be viewed as the percolation on the dual graph with $p = 0.49$.



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Theorem (Kesten, ...)

For $p \leq 0.5$, with probability 1 there is no infinite cluster in an (edge) percolation on $\mathbb{Z}^2$. For $p > 0.5$, with probability 1 there is such a cluster.

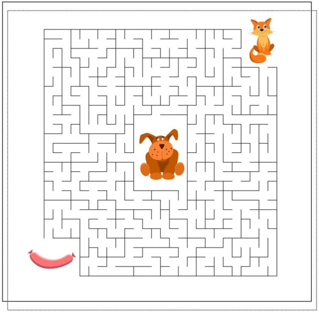

Figure: If we generate a maze randomly, what are the chances that the cat belongs to the same cluster with the sausage, but not the dog?

We will use the notation like $\mathbf{P}(ad|b|c)$ meaning the probability, in this case, that vertices a and d belong to the same cluster, which is different from the clusters of b and c.

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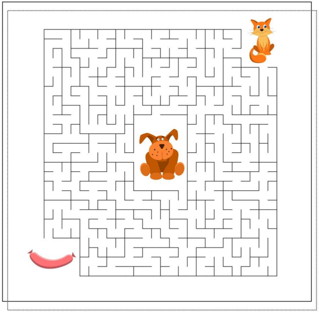

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$\mathbf{P}(CDS)$, $\mathbf{P}(CS|D)$, $\mathbf{P}(C|D|S)$, $\mathbf{P}(C|SD)$, $\mathbf{P}(CD|S)$.

The simplest question I am interested in is

Question

What are the possible values of these probabilities for all possible graph percolations?

The restrictions come in forms of inequalities, the most prominent of which is the Harris–Kleitman inequality (sometimes called the FKG inequality).
In particular, it prohibits all 5 probabilities to be equal to $\frac{1}{5}$.

Denote by H_n the n-dimensional discrete hypercube. We say that measure μ on H_n is a *product measure* if there exist probability measures μ_1, μ_2, ..., μ_n on $\{0, 1\}$, such that μ coincides with the direct product $\mu_1 \times \mu_2 \times \cdots \times \mu_n$. So, a percolation gives us a measure on H_n, where $n = |E|$.

Theorem (Harris–Kleitman inequality)

Let μ be a probability product measure on H_n, and $\mathcal{A}$ and $\mathcal{B}$ are events closed upwards. Then

$$\mathbf{P}(\mathcal{A} \cap \mathcal{B}) \geq \mathbf{P}(\mathcal{A})\mathbf{P}(\mathcal{B}).$$

Corollary

If $\mathcal{A}$ is closed upwards and $\mathcal{B}$ is closed downwards,

$$\mathbf{P}(\mathcal{A} \cap \mathcal{B}) \leq \mathbf{P}(\mathcal{A})\mathbf{P}(\mathcal{B}).$$

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Corollary

$$\mathbf{P}(CSD) \geq \mathbf{P}(CS)\mathbf{P}(DS).$$

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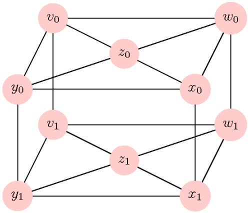
or

$$\mathbf{P}(CSD)\mathbf{P}(C|S|D) \geq \mathbf{P}(CS|D)\mathbf{P}(DS|C) + \mathbf{P}(CS|D)\mathbf{P}(DC|S)$$

Theorem (Gladkov)

$$\begin{aligned}&\mathbf{P}(CSD)\mathbf{P}(C|S|D)\\&\quad \geq \mathbf{P}(CS|D)\mathbf{P}(DS|C) + \mathbf{P}(CS|D)\mathbf{P}(DC|S) + \mathbf{P}(DS|C)\mathbf{P}(DC|S)\end{aligned}$$



Conjecture (Bunkbed conjecture)

A bunkbed graph G_b consists of two isomorphic graphs G, called the upper and lower bunks, and some additional edges, called posts; each post connects a vertex in the upper bunk with the corresponding isomorphic vertex in the lower bunk. We assign a probability to each edge, with each edge in the upper bunk assigned the same probability as the corresponding isomorphic edge in the lower bunk. The probabilities on the posts are arbitrary. The Bunkbed Conjecture states that in the percolated subgraph the probability that a vertex x in the upper bunk is connected to some vertex y in the upper bunk is greater than or equal to the probability that x is connected to y', the isomorphic copy of y in the lower bunk.

Remark

The conjecture follows from its partial case where all posts have probability 0 *or* 1.

Proof.

Indeed, $\mathbf{P}_{G_b}(xy)$ and $\mathbf{P}_{G_b}(xy')$ are polynomials in p_e. If e is a post, $\mathbf{P}_{G_b}(xy) - \mathbf{P}_{G_b}(xy')$ is linear in p_e, so we can move it to 0 or 1, depending on the sign of the coefficient in it. □

We call vertices with posts *transversal*.

Proposition

If there is only one transversal vertex v, the bunkbed conjecture is true.

Proof.

We can rewrite probabilities on G_b in terms of probabilities on G. So,

$$\mathbf{P}_{G_b}(xy) = \mathbf{P}_G(xy) \text{ and } \mathbf{P}_{G_b}(xy') = \mathbf{P}_G(xv)\mathbf{P}_G(yv) \leq \mathbf{P}_G(xyv) \leq \mathbf{P}_G(xy).$$

□

Theorem (van den Berg–Haggström–Kahn)

$$\mathbf{P}(a \leftrightarrow b \text{ and } c \leftrightarrow d \mid a \not\leftrightarrow d) \leq \mathbf{P}(a \leftrightarrow b \mid a \not\leftrightarrow d) \cdot \mathbf{P}(c \leftrightarrow d \mid a \not\leftrightarrow d);$$

so, conditionally on a and d are in different clusters, the events $a \leftrightarrow b$ and $c \leftrightarrow d$ are negatively correlated.

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Theorem (Alternative formulation)

$$\mathbf{P}(ab|cd)\mathbf{P}(a|d) \leq \mathbf{P}(ab|d)\mathbf{P}(a|cd)$$

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Proof.

We run a Markov chain process with a stable distribution being the uniform measure on $a|d$. Then we apply the Harris–Kleitman inequality to the events ab and cd which turn out to be closed upwards and downwards in the new coordinates. □

Proposition (Gladkov, Z.)

If there are only two transversal vertices v, w, the bunkbed conjecture is true.

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Proof.

Add together some Harris–Kleitmans and van den Berg–Haggström–Kahns.

$$
\begin{aligned}
&\mathbf{P}_{G_b}(xy) - \mathbf{P}_{G_b}(xy') = \\
&\qquad \mathbf{P}(xy|v|w) + \mathbf{P}(xy|vw) \\
&\qquad + \mathbf{P}((xv \cup xw) \cap (yv \cup yw)) - \mathbf{P}(xv \cup xw)\mathbf{P}(yv \cup yw) \\
&\qquad + \mathbf{P}(xv|w)\mathbf{P}(yw|v) - \mathbf{P}(xv|yw)\mathbf{P}(v|w) \\
&\qquad + \mathbf{P}(xw|v)\mathbf{P}(w|yv) - \mathbf{P}(xw|yv)\mathbf{P}(v|w)
\end{aligned}
$$

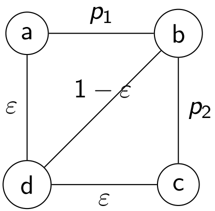
□

Question

What about 3 transversal vertices?

Question (Nikita)

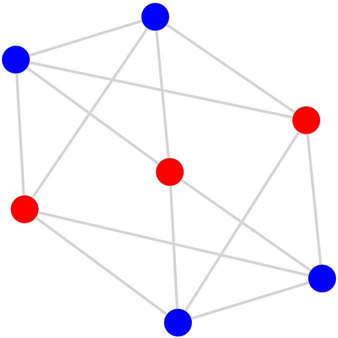
Can it be proved that if $\mathbf{P}(ac|b) \approx 0$, *then* $\mathbf{P}(abc) \approx \mathbf{P}(ab)\mathbf{P}(bc)$?



Question (Aleksandr)

Can it be proved that if $\mathbf{P}(ac|b) \approx \mathbf{P}(ab|c) \approx \mathbf{P}(a|bc) \approx 0$*, than* $\mathbf{P}(abc)$ *or* $\mathbf{P}(a|b|c)$ *is also* ≈ 0*?*

In particular, it is interesting if $\min(\mathbf{P}(abc), \mathbf{P}(a|b|c))$ is separated from $\frac{1}{2}$. The biggest minimum we can achieve is 0.29 on the graph in the Figure below where each red-blue edge has probability 0.32537 and both blue-blue edges have probability 0.19231. This way we get $\mathbf{P}(abc) \approx \mathbf{P}(a|b|c) \approx 0.29065$.



Definition

For two events $\mathcal{A}, \mathcal{B} \subseteq \Omega$, their *disjoint occurrence* $\mathcal{A} \,\square\, \mathcal{B}$ is defined as the event consisting of configurations x whose memberships in $\mathcal{A}$ and in $\mathcal{B}$ can be verified on disjoint subsets of indices. Formally, $x \in \mathcal{A} \,\square\, \mathcal{B}$ if there exist subsets $I, J \subseteq [n]$ such that:

- $I \cap J = \varnothing$,
- for all y that agrees with x on I (in other words, $y_i = x_i$ for all $i \in I$), y is also in $\mathcal{A}$, and
- similarly, every z that agrees with x on J is in $\mathcal{B}$.

Theorem (van den Berg–Kesten (vdBK))

$$\mathbf{P}(\mathcal{A} \,\square\, \mathcal{B}) \leq \mathbf{P}(\mathcal{A})\mathbf{P}(\mathcal{B})$$

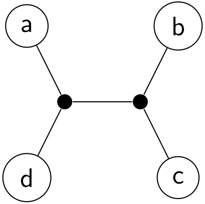
for every pair of closed upwards events $\mathcal{A}$ and $\mathcal{B}$.

Lemma (Hutchcroft)

$$\mathbf{P}(abcd) \leq \mathbf{P}(ab \cup ac \cup ad \cup bc \cup bd \cup cd)^2.$$

Proof.

Imagine that a, b, c, d are in one cluster. Then we can take a spanning tree of this cluster and find two nonintersecting path between a, b, c, d in it. Finally, we apply the vdBK inequality.



□

Corollary

$\min(\mathbf{P}(abcd), \mathbf{P}(a|b|c|d))$ *is less than the root of* $x = (1-x)^2$ *equal to* $\frac{3-\sqrt{5}}{2} \approx 0.38$.

Example (Decision tree techniques example)

Suppose I take cards from a shuffled deck one by one, until I get a spade. Then I take one more card. What are the chances that it is also a spade?

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Suppose I take cards from a shuffled deck one by one, until I get a spade. Then I take one more card. What are the chances that it is also a spade?

Solution: It is $\frac{1}{4}$, since we can invert the deck after the first spade without affecting the probability distribution. Under this transformation, the needed probability turns into a probability that the last card in the deck is a spade.

Definition

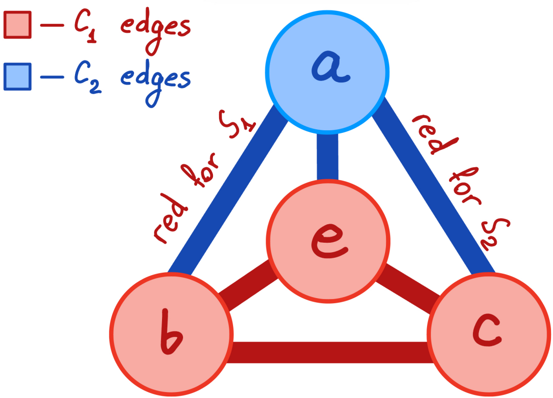
For two configurations $C_1, C_2 \in \Omega = 2^{[E]}$ and a set $S \subseteq E$ we denote by $C_1 \rightarrow_S C_2$ the configuration which coincides with C_1 on S and C_2 on its complement $\overline{S}$.

Lemma

Consider two independent bond Bernoulli percolations C_1 and C_2 having the same distribution μ on the same graph G. Let a decision tree T select each edge and reveal it in both C_1 and C_2. Furthermore, allow on each step, before revealing, decide if this edge will go to the set S (thus dependent on C_1 and C_2) or to its complement $\overline{S}$. Then $C_1 \rightarrow_S C_2$ is independent of $C_1 \rightarrow_{\bar{S}} C_2$ and both of them are distributed as μ.

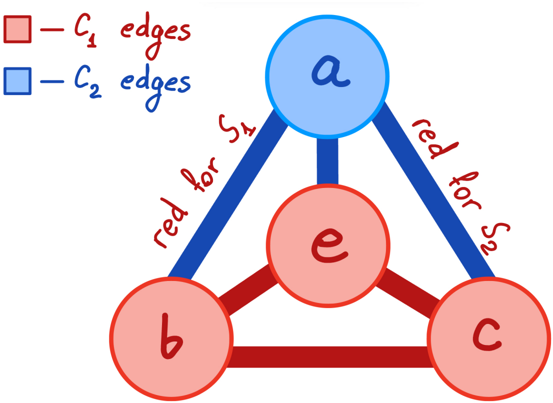
Consider the following decision trees S_1, S_2, and S_3 queuing the components of the graphs C_1 containing vertices a, b, c in the following order:

- S_1: b (taking edges from C_1), a (from C_2), and c (from C_1);
- S_2 is symmetric to S_1 by swapping b and c;
- S_3: a (taking edges from C_2), b and c (from C_1).



The key observation will be that not all possible configurations of $C_1 \to_{S_k} C_2 = p_k$ and $C_1 \to_{\overline{S_k}} = \overline{p_k}$ are possible. For example,

- if $C_1 \in a|b|c$, then $C_1 \to_{\overline{S_3}} C_2 \in a|b \cap a|c$;
- moreover, if in addition $C_1 \to_{S_3} C_2 \in ab \cup ac$, one has $C_1 \to_{S_1} C_2 \in ac$ or $C_1 \to_{S_2} C_2 \in ab$.



Proposition (Linear programming argument)

Consider the collection of potentials $\varphi_k(p_k, \overline{p_k})$ *satisfying*

$$\sum_k \varphi_k(p_k, \overline{p_k}) \geq 0 \text{ for all implementable } C_1 \to_{S_k} C_2 = p_k,\ C_1 \to_{\overline{S_k}} C_2 = \overline{p_k}.$$

Then, any possible collection of percolation probabilities $\mu = (\mu_{a|b|c}, \mu_{a|bc}, \mu_{b|ac}, \mu_{c|ab}, \mu_{abc})$ *satisfies the constraint*

$$\sum_{k,p,\overline{p}} \varphi_k(p, \overline{p}) \cdot \mu_p \cdot \mu_{\overline{p}} \geq 0.$$

Remark

If every collection of partitions $C_1 \to_{S_k} C_2 = p_k$, $C_1 \to_{\overline{S_k}} C_2 = \overline{p_k}$ *is implementable, then*

$$\sum_k \varphi_k(p, \overline{p}) \geq 0 \text{ for all } p, \overline{p},$$

and the inequality is trivial.

Theorem (Gladkov, Z.)

$$\mathbf{P}(a|b \cap a|c)\mathbf{P}(ab \cup ac) \leq \mathbf{P}(ab|c) + \mathbf{P}(ac|b) + \mathbf{P}(a|bc).$$

Corollary

$\mathbf{P}(abc)$ *and* $\mathbf{P}(a|b|c)$ *can not be simultaneously greater than* $(19 - 3\sqrt{37})/2 \approx 0.376$*.*

Thank you for your attention!

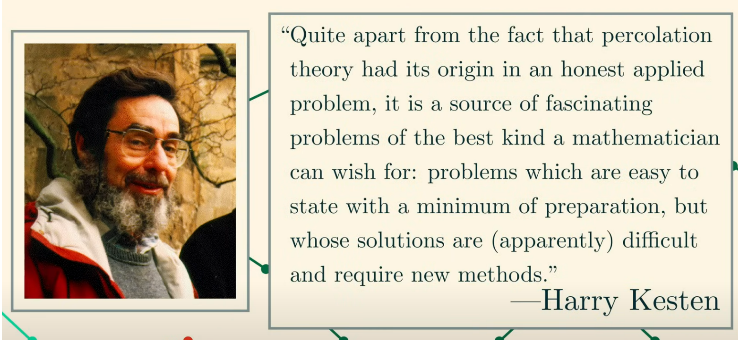


Figure: The critical probability for a site percolation on $\mathbb{Z}^2$ is around 0.592, so big QR-codes are unlikely to have a left to right path via black squares

In 1942, Rosalind Franklin, who then recently graduated in chemistry from the university of Cambridge, joined the BCURA. She started research on the density and porosity of coal. During the Second World War, coal was an important strategic resource. It was used as a source of energy, but also was the main constituent of gas masks.
Coal is a porous medium. To measure its 'real' density, one was to sink it in a liquid or a gas whose molecules are small enough to fill its microscopic pores. While trying to measure the density of coal using several gases (helium, methanol, hexane, benzene), and as she found different values depending on the gas used, Rosalind Franklin showed that the pores of coal are made of microstructures of various lengths that act as a microscopic sieve to discriminate the gases.

Conjecture (Kozma–Nitzan, 2024)

In a percolation on a graph having vertices a, b, $c_1, \ldots, c_n$ one has

$$\mathbf{P}(ab) \geq \mathbf{P}(ac_1 \cup ac_2 \cup \cdots \cup ac_n) \min_i \mathbf{P}(c_i b) \tag{1}$$

Theorem (Kozma–Nitzan)

Conjecture above implies that there is no infinite cluster in percolation on $\mathbb{Z}^d$ at a critical probability.

Proposition

$$\mathbf{P}(ab) \geq \mathbf{P}(ac_1 \cup ac_2)\Big(\frac{\mathbf{P}(ac_1|c_2)}{\mathbf{P}(ac_1|c_2)+\mathbf{P}(ac_2|c_1)}\mathbf{P}(c_1 b) + \frac{\mathbf{P}(ac_2|c_1)}{\mathbf{P}(ac_1|c_2)+\mathbf{P}(ac_2|c_1)}\mathbf{P}(c_2 b)\Big)$$

Question

What about 3 c_i's?