Lecture 2: Theory of Financial Risks

Jean-Philippe Bouchaud

by Shilong Yang
Department of Chemistry, MIT

April 12, 2001

1 Sum of iid Random Variables

People are particularly interested in constructing the distribution at all time scales according to the information on the short time scale, for example, daily or monthly price information. Under the assumption that all the fluctuations are identical independent distributed, which is, however, usually not justified, this reconstruction becomes possible. In this context, those stable distributions, such as Gaussian and Lévy distributions, play a special role. More generally, the distribution of the sum of iid variables will converge to one of the stable laws independent of the short-time distribution, which is the content of 'central limit theorem' (CLT). In practice, however, the convergence may be slow and thus of limited interest, in particular if one is concerned about intermediate time scales.

This sum of random variables in reality may represent the variation of the price of an asset within a month or a year, where the price variation in each day is assumed to be random and

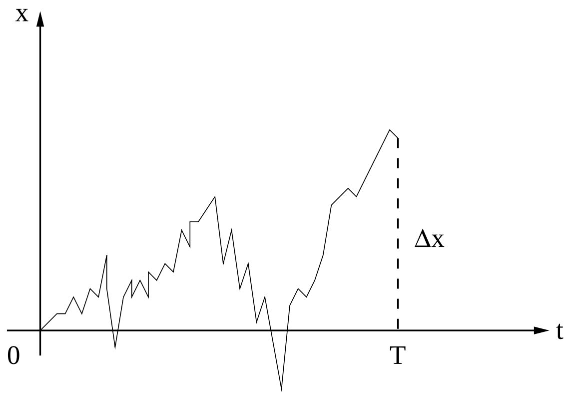

Figure 1: Example of the sum of iid random variables. Each step takes time τ.

independent. As illustrated in Fig. 1,

$$\Delta x = \sum_{i=1}^{N} \delta x_i \quad N = \frac{T}{\tau} \tag{1}$$

Assuming the PDF of the independent variable is $p_1(\delta x)$, then the PDF of the sum is obtained by convoluting the elementary distribution $p_1(\delta x)$ N times with itself, denoted as

$$p_N(\Delta x) = [p_1(\delta x)]^{\bigotimes N}. \tag{2}$$

The characteristic function of $p_1(\delta x)$ is the Fourier transform,

$$\hat{p}_1(k) = \int d\delta x p_1(\delta x) e^{ik\delta x}. \tag{3}$$

From the convolution theorem, the characteristic function of $p_N(\Delta x)$ is

$$\hat{p}_N(k) = [\hat{p}_1(k)]^N. \tag{4}$$

The moments and the cumulants are related to the characteristic function as

$$\hat{p}_1(k) = \int d\delta x p_1(\delta x) e^{ik\delta x} = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} m_n, \tag{5}$$

$$\log \hat{p}_1(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} c_n, \tag{6}$$

respectively. For Gaussian distribution, only the c_1 and c_2 are nonzero. From now on, we assume, $m_1 = 0$, i.e., $c_1 = 0$. From the expression of $\hat{p}_N(k)$, it is easy to prove that the cumulants are additive, i.e.,

$$C_{n,N} = NC_{n,1}, \text{ where } C_{n,1} = c_n. \tag{7}$$

Usually one likes to use the dimensionless parameters,

$$\lambda_{n,N} = \frac{C_{n,N}}{[C_{2,N}]^{n/2}} = \frac{C_{n,1}}{[C_{2,1}]^{n/2}} N^{1-\frac{n}{2}}, \text{ for } n > 2. \tag{8}$$

Especially, $\lambda_{3,N} \sim 1/\sqrt{N}$ is called the skewness, and $\lambda_{4,N} \sim 1/N$ is called the kurtosis.

Define $u = \Delta x/\sqrt{N}$,

$$\mathcal{P}_{>,N}(u_0) = \text{Prob. that } \frac{\Delta x}{\sqrt{N}} > u_0. \tag{9}$$

Therefore the intermediate asymtotic expansion of $\mathcal{P}_{>,N}(u)$ is

$$\begin{aligned} \mathcal{P}_{>,N}(u) &= \int_u^{\infty} p_G(u')du' + \frac{e^{-u^2}}{\sqrt{2\pi}}[\frac{Q_1(u)}{\sqrt{N}} + \frac{Q_2(u)}{N} + \cdots], \\ \text{where } Q_1(u) &= \frac{\lambda_{3,1}}{6}(u^2-1), \\ Q_2(u) &= \frac{\lambda_{4,1}}{24}(u^3-3u)|_{\lambda_{3,1}=0}. \end{aligned} \tag{10}$$

$p_G(u)$ is the PDF of Gaussian distribution function. In general, the N is not infinite large in reality. For example, τ is 30 minutes, and T is ten days, N can only be about 160. Therefore the tails in the distribution function is essentially important. Several cases are considered in the following.

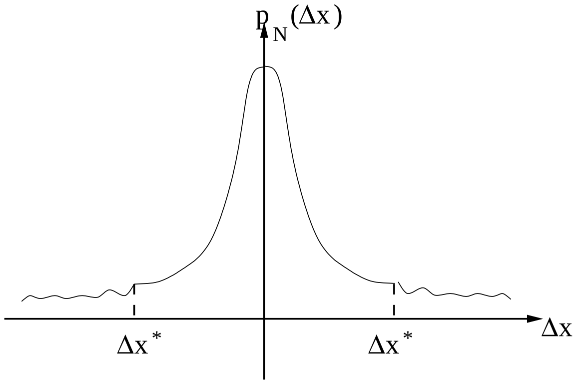

Figure 2: Demonstration of the edge of the "central region".

1. All cumulants are finite, and $\lambda_{3,1} = 0$ for simplicity.

   $u^* = \Delta x^*/\sqrt{N}$ charaterizes the edge of the central region, as shown in Fig. 2. So

$$\frac{\mathcal{P}_{>,N}(u^*) - \mathcal{P}_G(u^*)}{\mathcal{P}_G(u^*)} \approx \frac{e^{-u^{*2}/2}u^*/\sqrt{2\pi}}{e^{-u^{*2}/2}/\sqrt{2\pi}} \cdot \frac{\lambda_{4,1}}{24}u^{*3} \sim 1, \tag{11}$$

   where we apply the approximation

$$\mathcal{P}_G(u^*) = \frac{e^{-u^{*2}/2}}{u^*\sqrt{2\pi}}, \text{ for large } u^*. \tag{12}$$

   Therefore the edge of the central region is

$$\Delta x^* \sim \sqrt{N}\frac{N^{1/4}}{\lambda_{4,1}^{1/4}}. \tag{13}$$

2. Power-law tails

   Assuming the elementary distribution is a power-law function at large δx,

$$\mathcal{P}_1(\delta x) \simeq \frac{\mu A^\mu}{|\delta x|^{1+\mu}}, \text{ as } |\delta x| \to \infty. \tag{14}$$

   For the case $\mu > 2$, we can have a finite variance, i.e., $C_{2,1} < \infty$. The charateristic function is

$$\log \hat{p}(k) = \sum_{n=1}^{E(\mu)} \frac{(ik)^n}{n!}c_n + |k|^\mu c_\mu + \cdots \tag{15}$$

   It can be shown that

$$\begin{aligned} C_{\mu,N} &= NC_{\mu,1} \\ A_N^\mu &= NA_1^\mu \end{aligned} \tag{16}$$

by the similar argument as the cumulant expansion. So the edge of the "central region" is

$$\Delta x^* \sim \sqrt{N \log N}, \tag{17}$$

and the cumulative probability function at the edge is

$$\mathcal{P}_{>,N}(\Delta x^*) \sim \frac{1}{N^{\mu/2-1}(\log N)^{\mu/2}}. \tag{18}$$

For the case $\mu < 2$, the variance of the distribution is thus infinite. The limit distribution for large N is then a stable Lévy distribution of exponent μ and with a tail amplitude NA^μ.

2 Model of Weak Correlations

There are cases when the random variables are not identical and independently distributed. Here we assume only weak correlatios among these random variables. Specifically, the sum of these random variables can be expressed as

$$\begin{aligned}\Delta x &= \sum \delta x_i, \\ \delta x_i &= \sigma_i \epsilon_i,\end{aligned} \tag{19}$$

where ϵ_i are iid random variables with zero mean and unit variance, and σ_i are long-range correlated random variables. Therefore, ϵ_i is the sign of δx_i, while σ_i is the amplitude. We use $\langle \cdots \rangle$ to represent the average over the ϵ-noise, and $\overline{\cdots}$ to represent the average over the σ-noise. The correlations can be explicitly written as

$$\langle \epsilon_i \epsilon_j \rangle = \delta_{ij}, \tag{20}$$

$$c_{ij} = \overline{\sigma_i^2 \sigma_j^2} - \overline{\sigma_i^2}\,\overline{\sigma_j^2}, \tag{21}$$

where stationary correlation has be assumed for random variables σ_i. The correlation function of δx_i is thus given by

$$\overline{\langle \delta x_i \delta x_j \rangle} = \overline{\sigma_i \sigma_j} \langle \epsilon_i \epsilon_j \rangle = \overline{\sigma_i^2} \delta_{ij}. \tag{22}$$

Hence the δx_i are uncorrelated random variables, however, they are not independent since a higher order correlation reveals a richer structure. For example, the correlation function of δx_i^2 is

$$\overline{\langle \delta x_i^2 \delta x_j^2 \rangle} - \overline{\langle \delta x_i^2 \rangle \langle \delta x_j^2 \rangle} = \overline{\sigma_i^2 \sigma_j^2} - \overline{\sigma_i^2}\,\overline{\sigma_j^2}. \quad (i \neq j) \tag{23}$$

It is easy to show the variance and the kurtosis of $\Delta x = \sum \epsilon_i \sigma_i$

$$\begin{aligned}\overline{\langle \Delta x^2 \rangle} &= \sum_{i=1}^{N} \overline{\sigma_i^2} = N\overline{\sigma^2}, \\ \lambda_{4,N} &= \frac{\overline{\langle \Delta x^4 \rangle} - 3\overline{\langle \Delta x^2 \rangle}}{\overline{\langle \Delta x^2 \rangle}^2} \\ &= \frac{1}{N}[\kappa_\epsilon + (3 + \kappa_\epsilon) g(0) + 6 \sum_{k=1}^{N} (1 - \frac{l}{N}) g(l)],\end{aligned} \tag{24}$$

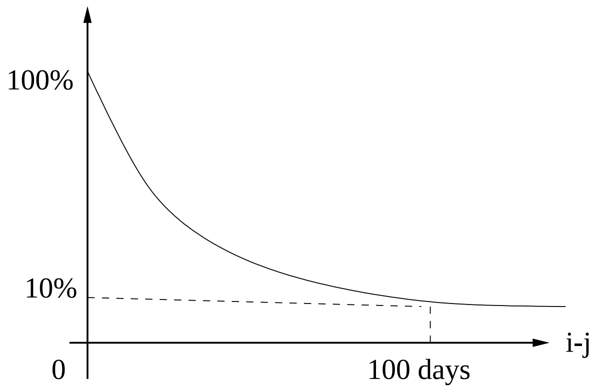

Figure 3: An example of a slow varying correlation function.

where $g(l)$ is the correlation function of the variance,

$$\overline{\sigma_i^2 \sigma_j^2} - \overline{\sigma^2}^2 = g(|i-j|)(\overline{\sigma^2})^2, \tag{25}$$

and κ_ϵ is the kurtosis of the ϵ-noise.It is interesting to see that for $N = 1$, the above formula gives $\lambda_{4,1} = \kappa_\epsilon + (3 + \kappa_\epsilon)g(0)$, which is greater than κ_ϵ. The expression of $\lambda_{4,1}$ indicates that even if $\kappa_\epsilon = 0$, a flutuating volatility is enough to produce nonzero kurtosis.More importantly, one sees that if the variance correlation $g(l)$ decays with l, the kurtosis $\lambda_{4,N}$ tends to zero with N, thus showing that the sum indeed converges toward a Gaussian variable. A typical long-range correlation function can be assumed to be a power law, $g_l \sim 1/l^\nu$, an example is shown in Fig. 3. If $\nu > 1$,

$$\lambda_{4,N} \propto \frac{1}{N}, \tag{26}$$

which is similar to the result of the sum of iid random variables. However, if $0 < \nu < 1$,

$$\lambda_{4,N} \propto \frac{1}{N^\nu}, \tag{27}$$

which decays extremely slowly. Hence the long-range correlation in the variance considerably slows down the convergence toards the Gaussian. For example, when $N = 1000$, $\nu = 0.3$, $\lambda_{4,N} \simeq 0.13$, which is more than 10%. Therefore the tails have a fairly large amount of distribution, which can not be ignored.