Lecture 15: Persistent Random Walk (Continued)

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1 Exact Markov Chain for the Persistent Random Walk (Review)

Consider the one-dimensional "persistent random walk" on the integers, in which the walker has a probability of α of stepping in the same direction as the previous step, and a probability $\beta = 1 - \alpha$ of reversing direction. In the previous lecture we showed that each step has a partial correlation of $\rho = 2\alpha - 1$ with the previous step, and that the correlation function is exponentially decaying $c(n) = \rho^n$, which implies diffusive scaling of the position with a modified diffusion constant

$$\frac{D}{D_o} = \frac{1+\rho}{1-\rho} = \frac{\alpha}{\beta} \tag{1.1}$$

where D_o is the "uncorrelated" diffusion constant when $\alpha = 1/2$ ($\rho = 0$).

In the previous lecture, we derive a pair of coupled difference equations

$$A_{n+1}(m) = \alpha A_n(m-1) + \beta B_n(m-1), \tag{1.2}$$

$$B_{n+1}(m) = \beta A_n(m+1) + \alpha B_n(m-1), \tag{1.3}$$

which is a "Markov chain" (because the next step only depends on the "state" at the current time, with no history dependence) where the "state vector"

$$A_n = \Pr\mathrm{ob}\,(X_n = m \,|\, X_n = m-1)\, Prob(X_n = m-1), \tag{1.4}$$

$$B_n = \Pr\mathrm{ob}\,(X_n = m \,|\, X_n = m+1)\, Prob(X_n = m+1), \tag{1.5}$$

is specified by the probabilities of being at m at time n having come from either the left, $A_n(m)$, or the right, $B_n(m)$. The initial condition that the walker is equally likely to move left or right in the first step,

$$A_0(m) = \frac{1}{2}\delta_{m,0}, \tag{1.6}$$

$$B_0(m) = \frac{1}{2}\delta_{m,0}, \tag{1.7}$$

where $\delta_{m,0}$ is the Kronecker delta function defined as

$$\delta_{m,n} = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases} . \tag{1.8}$$

2 Continuum Approximation of Long-Time Behavior

2.1 Continuum Expansions

Let $P_n(m) = A_n(m) + B_n(m) = \Pr(X_n = m)$ be the probability of being at m at time n, regardless of the previous step, and let $Q_n(m) = A_n(m) - B_n(m)$ be the excess occupation probability due to temporal correlations. Let $p(m\sigma, n\tau) = P_n(m)$ and $q(n\sigma, n\tau) = Q_n(m)$, where σ is the lattice spacing and τ is the time step. For $n \gg 1$, the probability will have spread enough compared to the lattice spacing to justify a long-wavelength continuum approximation, where the functions $p(x,t)$ and $q(x,t)$ in the difference equations above are well-approximated by Taylor expansions in the limit $\sigma, \tau \to 0$:

$$\tau p_t + \frac{\tau^2}{2} p_{tt} + \cdots = (\beta - \alpha)\,\sigma q_x + (\alpha - \beta)\,\frac{\sigma^2}{2} p_{xx} + \cdots\,, \tag{2.1}$$

$$q + \tau q_t + \frac{\tau^2}{2} q_{tt} + \cdots = (\alpha - \beta)\,q - \sigma p_x + (\alpha - \beta)\,\frac{\sigma^2}{2} q_{xx} + \cdots \tag{2.2}$$

(This approximation is justified rigorously below by solving the difference equations exactly.)

2.2 Diffusive Scaling

Here we will first consider the diffusive scaling $\sigma^2/2\tau \to D_0$ to arrive at a diffusion equation for the random walk, albeit with a different diffusion coefficient. Take the x-derivative of eq.(2.2),

$$q_x(1 - \alpha + \beta) + \tau q_{tx} + \frac{\tau^2}{2} q_{ttx} + \cdots = -\sigma p_{xx} + (\alpha - \beta)\,\frac{\sigma^2}{2} q_{xxx} + \cdots\,. \tag{2.3}$$

Noting that $1 - \alpha = \beta$, then in the limit of $\sigma, \tau \to 0$

$$q_x = -\frac{\sigma}{2\beta} p_{xx}. \tag{2.4}$$

Substitute the latter into (2.1) and take the limit $\sigma, \tau \to 0$ to arrive at the diffusion equation

$$p_t = D_0 \left(1 + \frac{\alpha - \beta}{\beta}\right) p_{xx} + \mathcal{O}(\tau). \tag{2.5}$$

To leading order, the latter is a diffusion equation with a modified diffusion coefficient,

$$\frac{D}{D_0} = 1 + \frac{\alpha - \beta}{\beta} = \frac{\alpha}{\beta} = \frac{\alpha}{1 - \alpha} = \frac{1 + \rho}{1 - \rho} \tag{2.6}$$

as generally expected for the correlation function $c(n) = \rho^n$.

2.3 Ballistic Scaling

Here we will instead use the ballistic scaling, $\sigma/\tau \to v$, in addition to letting $\alpha \to 1$ as $\sigma, \tau \to 0$. This provides the set of relationships

$$\alpha = 1 - \frac{\epsilon}{2}, \quad \beta = \frac{\epsilon}{2}, \quad \alpha - \beta = 1 - \epsilon, \tag{2.7}$$

where ϵ is a small parameter to be determined later from the continuum limit (i.e. such that the continuum limit does not yield $p = 0$). In the ballistic regime, eq. (2.1) has the asymptotic form

$$p_t \sim (\beta - \alpha)\,\frac{\sigma}{\tau} q_x = -\,(1 - \epsilon)\,v q_x, \tag{2.8}$$

whereby

$$p_t \sim -v q_x. \tag{2.9}$$

As in the diffusive case, take the x-derivative of (2.2) to arrive at (2.3), which may be rewritten in the form

$$\epsilon q_x + \tau q_{xt} + \cdots = -\sigma p_{xx} + (1 - \epsilon)\frac{\sigma^2}{2} q_{xxx} + \cdots . \tag{2.10}$$

Asymptotically,

$$q_x \sim -\frac{\tau}{\epsilon} q_{tx} - \frac{\sigma}{\epsilon} p_{xx}. \tag{2.11}$$

If $q_x = 0$, then eq. (2.9) is trivial in the continuum limit. In order to keep q_x nontrivial, define a correlation time (and hence a scale for the small parameter ϵ)

$$\frac{\tau}{\epsilon} \to \tau_c, \tag{2.12}$$

whereby (2.11) becomes

$$q_x \sim -\tau_c q_{tx} - v\tau_c p_{xx}. \tag{2.13}$$

To express q_{tx} in terms of p, differentiate (2.9) with respect to t,

$$p_{tt} \sim -v q_{tx}. \tag{2.14}$$

Substiting (2.14) into (2.13) and placing the result (2.9) yields the **telegrapher's equation**

$$p_{tt} + \frac{1}{\tau_c} p_t = v^2 p_{xx}, \tag{2.15}$$

at leading order as $\sigma, \tau \to 0$ and $\alpha \to 1$, with ballistic scalings $\sigma = vt$ and $\alpha = 1 - \tau/2\tau_c$.

For times small compared to the correlation time, then the telegrapher's equation reduces to the wave equation,

$$p_{tt} \sim v^2 p_{xx}, \quad t \ll \tau_c. \tag{2.16}$$

Provided that there exists some initial velocity, then the probability translates without spreading (the "signal"). For times long compared to the correlation time, the telegrapher's equation becomes the diffusion equation,

$$p_t \sim v^2 \tau_c p_{xx}, \tag{2.17}$$

consistent with the diffusive scaling limit above. This transition in the telegrapher's equation represents spreading of the probability, relative to translation, which corresponds to "noise" growing in the signal.

3 Exact Solution of Difference Equations

In this section, we will solve the difference equations (1)-(2) exactly using discrete Fourier transforms (Fourier series), e.g.

$$\widehat{A}_n(k) = \sum_{m=-\infty}^{\infty} \exp(imk) A_n(m), \tag{3.1}$$

and its inverse,[1]

$$A_n(m) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \exp(-imk) \widehat{A}_n(k) dk. \tag{3.2}$$

The transform of the difference equations (1.2) and (1.3) is then, in vector form,

$$\begin{pmatrix} \widehat{A}_{n+1}(k) \\ \widehat{B}_{n+1}(k) \end{pmatrix} = \begin{pmatrix} \alpha \exp(ik) & \beta \exp(ik) \\ \beta \exp(-ik) & \alpha \exp(-ik) \end{pmatrix} \begin{pmatrix} \widehat{A}_n(k) \\ \widehat{B}_n(k) \end{pmatrix}, \tag{3.3}$$

since the shifts in time become angular shifts in k-space.

The exact solution for the transform of the walk after n steps (similar to the geometrically decaying steps) is of the form

$$\begin{pmatrix} \widehat{A}_n(k) \\ \widehat{B}_n(k) \end{pmatrix} = \begin{pmatrix} \alpha \exp(ik) & \beta \exp(ik) \\ \beta \exp(-ik) & \alpha \exp(-ik) \end{pmatrix}^n \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \tag{3.4}$$

where we have made use of the transform of the initial condition. The solution of this equation will ultimately require computing the eigenvalues and eigenvectors of the coefficient matrix.

To simplify the subsequent analysis, note the system of eigenvalue equations for a matrix $\mathbf{M}$ can be written in the matrix form

$$\mathbf{MS} = \mathbf{S}\Lambda, \tag{3.5}$$

where $\mathbf{S}$ is the matrix whose columns are the eigenvectors and Λ is the (diagonal) matrix of the corresponding eigenvalues. Post-multiply both sides by $\mathbf{S}^{-1}$,

$$\mathbf{M} = \mathbf{S}\Lambda\mathbf{S}^{-1}. \tag{3.6}$$

[1] As an aside, these transforms are related to the "z-transforms" commonly used in electrical engineering,

$$\widetilde{A}_n(z) = \sum_{m=-\infty}^{\infty} A_n(m) z^m, \quad A_n(m) = \frac{1}{2\pi i} \oint \frac{\widetilde{A}_n(z) dz}{z^{m+1}}$$

which amounts to viewing the discrete probabilities as coefficients in a Laurent series in the complex plane. On the unit circle $z = e^{ik}$ (k real), this Laurent series is equivalent to the Fourier series, $\widetilde{A}_n(e^{ik}) = \widehat{A}_n(k)$. The z-transform is called the "probability generating function" in the mathematical literature. In that case, the moments are conveniently related to the asymptotic properties of the generating function at $z = 1$. For example, $1 = \sum A(m) = \tilde{A}(1)$, mean $= \langle m \rangle = \sum m A(m) = d\tilde{A}/dz(1)$, variance $= \langle m^2 \rangle - \langle m \rangle^2 = \sum (m-1) m A(m) = d^2\tilde{A}/dz^2(1)$, etc.

Raising both sides to the n power,

$$\mathbf{M}^n = \left(\mathbf{S}\Lambda\mathbf{S}^{-1}\right)^n = \mathbf{S}\Lambda^n\mathbf{S}^{-1} \tag{3.7}$$

Computation of the eigenvalues λ requires the solution of the algebraic equation

$$\det(\lambda\mathbf{I} - \mathbf{M}) = 0, \tag{3.8}$$

with $\mathbf{I}$ the identity matrix. The latter furnishes the quadratic equation,

$$[\alpha\exp(ik) - \lambda]\,[\alpha\exp(-ik) - \lambda] - \beta^2 = 0, \tag{3.9}$$

which has the simpler form,

$$\lambda^2 - 2\lambda\alpha\cos(k) + \left(\alpha^2 - \beta^2\right) = 0, \tag{3.10}$$

upon use of the identity $2\cos(k) = \exp(ik) + \exp(-ik)$. The solution to the quadratic equation is

$$\lambda_\pm = \alpha\cos(k) \pm \sqrt{\alpha^2\cos^2(k) - (\alpha^2 - \beta^2)} \tag{3.11}$$

Using the identity $\sin^2 k + \cos^2 k = 1$, this takes the simple form

$$\lambda_\pm = \alpha\cos(k) \pm \sqrt{\beta^2 - \alpha^2\sin^2(k)}. \tag{3.12}$$

For asymptotically small values of k, the eigenvalues are

$$\lambda_\pm \sim \alpha\left(1 - \frac{k^2}{2}\right) \pm \beta\left[1 - \left(\frac{\alpha k}{2\beta}\right)^2\right]. \tag{3.13}$$

where we have made use of the asymptotic expansion

$$\sqrt{1-\epsilon^2} \sim 1 - \frac{\epsilon^2}{2} + \cdots. \tag{3.14}$$

Equation (3.13) can be rewritten as

$$\lambda_\pm \sim (\alpha \pm \beta) - \frac{\alpha k^2}{2\beta}(\beta \pm \alpha). \tag{3.15}$$

The plus eigenvalue, upon noting that $\alpha + \beta = 1$, reduces to

$$\lambda_+ \sim 1 - \frac{\alpha k^2}{2\beta}. \tag{3.16}$$

Define $w = k\sqrt{n}$ and multiply (3.16) by n/n to arrive at

$$\lambda_+^n \sim \exp\left(-\frac{\alpha w^2}{2\beta}\right), \tag{3.17}$$

with $n \to \infty$ with w fixed. The latter result requires use of the definition

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \exp(x). \tag{3.18}$$

Using similar techniques, the minus eigenvalue is

$$\lambda_-^n \sim (\alpha-\beta)^n \left(1+\frac{\alpha k^2}{2\beta}\right)^n, \tag{3.19}$$

which, upon using (3.18), reduces to

$$\lambda_-^n \sim (\alpha-\beta)^n \exp\left[\frac{\alpha w^2}{2\beta}\right]. \tag{3.20}$$

Noting that the exponential is finite and that $(\alpha-\beta) < 1$, then, as $n \to \infty$ with w fixed,

$$\lambda_-^n \to 0. \tag{3.21}$$

The eigenvectors are then computed as the nontrivial solutions $\mathbf{x}$ of the expression

$$(\lambda\mathbf{I}-\mathbf{A})\,\mathbf{x} = \mathbf{0}. \tag{3.22}$$

The resulting algebraic equation for the components of the eigenvector is

$$[\alpha\exp(ik)-\lambda_\pm]\,x_1 + \beta\exp(ik)x_2 = 0, \tag{3.23}$$

where x_1 and x_2 are the components of the eigenvector. Using the $k \to 0$ limit, substitution of $\lambda_+ \sim 1$ furnishes the asymptotic expression

$$(\alpha-1)\,x_1 + \beta x_2 \to 0, \tag{3.24}$$

which is satisfied non-trivially by the choice

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3.25}$$

noting again that $\alpha+\beta = 1$. For the asymptotic value of the negative eigenvalue, $\lambda_- \to 0$, (3.23) is of the form

$$\alpha x_1 + \beta x_2 \sim 0, \tag{3.26}$$

which is satisfied non-trivially by the choice

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}. \tag{3.27}$$

Accordingly, we identify the following matrices,[2]

$$\mathbf{S} = \begin{pmatrix} 1 & -\beta \\ 1 & \alpha \end{pmatrix}, \quad \mathbf{S}^{-1} = \begin{pmatrix} \alpha & \beta \\ -1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.28}$$

[2]Note that $\det\mathbf{S} = \alpha+\beta = 1$.

From (3.7),

$$\mathbf{M}^n \sim \begin{pmatrix} 1 & -\beta \\ 1 & \alpha \end{pmatrix} \begin{pmatrix} \lambda_+^n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -1 & 1 \end{pmatrix} \sim \lambda_+^n \begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}, \tag{3.29}$$

as $n \to \infty$ with w fixed.

Employing the latter asymptotic expression, as well as (3.17), in (3.4), the asymptotic solution for the walk is

$$\begin{pmatrix} \widehat{A}_n(k) \\ \widehat{B}_n(k) \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \exp\left(-\frac{\alpha w^2}{2\beta}\right), \tag{3.30}$$

since $\alpha + \beta = 1$. We know then that the transform of the difference between the conditional probabilities $Q_n(m)$ (and hence the local effect of correlations) vanishes,

$$\widehat{Q}_n = \widehat{A}_n(k) - \widehat{B}_n(k) \to 0. \tag{3.31}$$

(The global effect of correlations, however, is not negligible, as evidenced by a changed diffusion coefficient given below.)

As in the continuum limit, the transform of the probability is just the sum of the left and right steps,

$$\widehat{P}_n(k) = \widehat{A}_n(k) + \widehat{B}_n(k), \tag{3.32}$$

then it has the explicit value

$$\widehat{P}_n(k) \sim \exp\left(-\frac{\alpha w^2}{2\beta}\right). \tag{3.33}$$

Substitute back for k in the latter and invert to arrive at the Gaussian form (i.e. the "CLT" holds),

$$P_n(m) \sim \left(2\pi\frac{\alpha n}{\beta}\right)^{-1/2} \exp\left(-\frac{m^2}{2\alpha n/\beta}\right), \tag{3.34}$$

as $n \to \infty$, $\zeta = m/\sqrt{n}$ fixed. Finally, note that the diffusion coefficient, $D/D_0 = \alpha/\beta$, is identical to that derived in the continuous case (2.6).