Supplementary Note on Lecture 16: Exact Probability Distribution for the Bernoulli Random Walk

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Consider the probability distribution for the Bernoulli random walk, given by the Fourier-type integral

$$P_n(m) = \int_{-\pi}^{\pi} \frac{dk}{2\pi}\, e^{-imk}\, \cos^n(k), \qquad m \in \mathbb{Z}, \quad n \in \mathbb{N}. \tag{1}$$

This integral was evaluated in class via the steepest descents method in the physically significant case of large n ($n \gg 1$) and fixed $\xi = m/n$. In this note, $P_n(m)$ is calculated exactly (for all admissible m and n) by contour integration. Notice that

$$P_n(-m) = \int_{\pi}^{-\pi} \frac{d(-k)}{2\pi}\, e^{-imk}\, \cos^n(-k) = P_n(m), \tag{2}$$

i.e., $P_n(m)$ is even in m. Hence, without loss of generality, assume m to be non-negative, $m \geq 0$, unless it is stated otherwise. Distinguish the following cases:

(i) $\xi = m/n > 1$.

Let $m = n + p + 1$, where p is a *non-negative* integer. The change of variable $\zeta = e^{-ik}$, $k \in [-\pi, \pi)$, recasts integral (1) in the form

$$P_n(m) = -\oint_C \frac{d\zeta/(-i\zeta)}{2\pi}\, \zeta^m \left(\frac{\zeta + \zeta^{-1}}{2}\right)^n = \frac{2^{-n}}{2\pi i} \oint_C d\zeta\, \zeta^p\, (1+\zeta^2)^n, \tag{3}$$

where C henceforth denotes the unit circle $\{\zeta \in \mathbb{C} :\ |\zeta| = 1\}$ traversed in the counterclockwise (positive) sense. Clearly, the integrand is an analytic function of ζ. According to the Cauchy integral theorem,

$$P_n(m) = 0, \qquad m > n. \tag{4}$$

(ii) $m = n$, i.e., $\xi = 1$.

The steps that led to (3) now lead to a contour integral with an integrand that exhibits a simple pole at $\zeta = 0$:

$$\frac{1}{\zeta}(1+\zeta^2)^n = \frac{1}{\zeta} + \mathcal{O}(\zeta), \qquad \zeta \to 0. \tag{5}$$

The Cauchy integral formula furnishes the anticipated formula

$$P_n(n) = \frac{2^{-n}}{2\pi i}\; 2\pi i\, \mathrm{Res}_{\zeta=0}\left\{\frac{1}{\zeta}(1+\zeta^2)^n\right\} = 2^{-n}. \tag{6}$$

(iii) $\xi = m/n < 1$.

This range of values includes the case examined in class. Let $n = m + p + 1$, where p is a *non-negative* integer. With $\zeta = e^{ik}$, integral (1) reads as

$$P_n(m) = \frac{2^{-n}}{2\pi i}\oint_C d\zeta\; \zeta^p\,(1+\zeta^{-2})^n. \tag{7}$$

The integrand here is analytic for $|\zeta| > \epsilon, \epsilon > 0$, but may have a pole at $\zeta = 0$. The use of the binomial expansion

$$(1+\zeta^{-2})^n = \sum_{l=0}^{n} \frac{n!}{(n-l)!\; l!}\; \zeta^{-2l} \tag{8}$$

under the integral sign, interchange of summation and integration, and application of the Cauchy integral formula yield

$$P_n(m) = \frac{2^{-n}}{2\pi i}\sum_{l=0}^{n}\frac{n!}{(n-l)!\; l!}\oint d\zeta\; \zeta^{p-2l} = 2^{-n}\sum_{l=0}^{n}\frac{n!}{(n-l)!\; l!}\;\delta_{n-m,2l}. \tag{9}$$

Note that this interchange is legitimate to all orders in n, since C can be deformed to a larger circle in (7) and the expansion then converges as $n \to \infty$. It follows that

$$P_n(m) = 2^{-n}\,\frac{n!}{\left(\frac{n-m}{2}\right)!\;\left(\frac{n+m}{2}\right)!}, \qquad n-m:\ \text{even}, \tag{10}$$

or,

$$P_n(m) = 0, \qquad n-m:\ \text{odd}. \tag{11}$$

In compact notation,

$$P_n(m) = [1+(-1)^{n+m}]\,2^{-n-1}\,\frac{n!}{\Gamma\left(1+\frac{n-m}{2}\right)\;\Gamma\left(1+\frac{n+m}{2}\right)}, \qquad n > m, \tag{12}$$

where the factor in front takes care of (11), and $\Gamma(z)$ is the gamma function defined by the integral

$$\Gamma(z) = \int_0^\infty dt\, e^{-t}\, t^{z-1}, \qquad \mathrm{Re}\ z > 0. \tag{13}$$

The ensuing formula $\Gamma(1+z) = z\Gamma(z)$ implies that the analytic continuation of $\Gamma(z)$ to $\mathrm{Re}\ z \leq 0$ has simple poles at $z = -j,\ j \in \mathbb{N}$.

Let us now show that (12) gives the correct results even when $m \geq n$. Indeed, for $m = n$ its RHS gives

$$[1 + (-1)^{2n}]\, 2^{-n-1}\, \frac{n!}{\Gamma(1)\ \Gamma(1+n)} = 2^{-n},$$

in agreement with (6). For $m > n$ one needs to consider $m - n$ as odd or even separately. In the former case, $[1 + (-1)^{n+m}] = 0$ and (4) follows. In the latter case, $1 + (n-m)/2 \equiv w$ is a negative integer or zero, $\Gamma(w) \to \infty$ and (4) follows again. In conclusion, by letting m take all integer values,

$$P_n(m) = [1 + (-1)^{n+m}]\, \frac{2^{-n-1}\, n!}{\Gamma\left(1 + \dfrac{n - |m|}{2}\right)\ \Gamma\left(1 + \dfrac{n + |m|}{2}\right)}, \qquad m \in \mathbb{Z},\ m \in \mathbb{N}. \tag{14}$$

It remains to write down an approximation to $P_n(m)$ for $n \gg 1$ which is uniform in $\xi = m/n$. In view of the familiar Stirling formula

$$\Gamma(1+v) \sim \sqrt{2\pi v}\, e^{v \ln v - v}, \qquad v \to \infty,\ |\mathrm{Arg}\ v| < \pi, \tag{15}$$

Eq. (14) yields

$$\begin{aligned} P_n(m) \quad \sim \quad & [1 + (-1)^{n+m}]\, \frac{2^{-n-1/2}}{\sqrt{1+|\xi|}}\, \frac{\exp\left[\dfrac{1-|\xi|}{2}\,(n \ln n - n)\right]}{\Gamma\left(1 + \dfrac{1-|\xi|}{2}\, n\right)} \\ & \times \exp\left[-n\, \frac{1+|\xi|}{2}\ \ln\left(\frac{1+|\xi|}{2}\right)\right], \qquad n \gg 1, \end{aligned} \tag{16}$$

uniformly in $\xi, -\infty < \xi < +\infty$. It should be borne in mind however that ξ is a rational number such that the argument of the gamma function in the denominator is an integer or a half-integer. It is left as an exercise to show that this formula reduces to the known steepest descents result if

$$\frac{1-|\xi|}{2}\, n \gg 1, \tag{17}$$

so that Stirling's formula is applicable once more.