Lecture 17: Steepest Descent Asymptotics for Fat-Tailed Walks

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In the last lecture we introduced the method of Steepest Descent, and applied it to the Bernoulli random walk, as an example first case. In this lecture, we study a walk with finite variance, but divergent higher-order moments. We use the method of Steepest Descent to derive an approximation for the probability distribution of the walkers, that is valid both in the central region and in the tails.

1 Student-t distribution

We study the iid random walk with transition probability distribution

$$p(x) = \frac{2/\pi}{(1+x^2)^2}. \tag{1}$$

This is the student-t distribution. Although the variance is finite, $< |x|^3 > = \infty$, and all higher-order moments diverge. The structure function was found in a previous homework problem to be

$$\hat{p}(k) = e^{-|k|}(1+|k|) \tag{2}$$

It was also shown that, although $\hat{p}$ is not analytic at the origin, it does exhibit k^2 behavior at the leading order:

$$\hat{p}(k) \sim 1 - \frac{k^2}{2} + O(|k|^3) \tag{3}$$

This leading-order behavior leads to the conclusion that, in the limit $n \to \infty$, the Central Limit Theorem holds, and the distribution of random walkers is normal.

Since the steps are iid, the distribution $P_n(x)$ for the walker after n steps is given by the inverse transform

$$P_n(x) = \int_{-\infty}^{\infty} e^{ikx} e^{-n|k|}(1+|k|)^n \frac{dk}{2\pi} \tag{4}$$

We may manipulate this expression in the following way. Using Euler's formula, $e^{ikx} = cos\ kx + i\ sin\ kx$ gives

$$P_n(x) = \int_{-\infty}^{\infty} (cos\ kx + i\ sin\ kx) e^{-n|k|}(1+|k|)^n \frac{dk}{2\pi}. \tag{5}$$

Since sin is odd, and $e^{-n|k|}(1+|k|)^n$ is even, then the sin term drops out, to give

$$P_n(x) = \int_{-\infty}^{\infty} cos\ kx\ e^{-n|k|}(1+|k|)^n \frac{dk}{2\pi} \tag{6}$$

and since cos is even, we may replace the integral over $(-\infty, \infty)$ by twice the integral over the positive real axis. This also allows us to drop the absolute value signs.

$$P_n(x) = \frac{1}{\pi}\int_{0}^{\infty} cos\ kx\ e^{-nk}(1+k)^n dk \tag{7}$$

and finally noting that $cos\ kx = Re(e^{ikx})$, and that the rest of the integrand is real, we have

$$P_n(x) = \frac{1}{\pi} Re\left\{\int_{0}^{\infty} e^{ikx} e^{-nk}(1+k)^n dk\right\} \tag{8}$$

This expression may be written in the form of a Laplace integral,

$$P_n(x) = \frac{1}{\pi} Re\left\{\int_{0}^{\infty} e^{-n\psi(k,\xi)} dk\right\} \tag{9}$$

$$\psi(k,\xi) = -i\xi k + k - log(1+k) \tag{10}$$

$$\xi = \frac{x}{n} \tag{11}$$

Since we wish to use the method of Steepest Descent with the integral (9), we need to study the critical point(s) of ψ, which correspond to the saddle points for the Laplace integrand. The first two k-derivatives of ψ are given by

$$\frac{\partial\psi}{\partial k} = 1 - i\xi - \frac{1}{1+k} \tag{12}$$

$$\frac{\partial^2\psi}{\partial k^2} = \frac{1}{(1+k)^2} \tag{13}$$

Setting $\partial\psi/\partial k = 0$ and solving for the critical point k_s,

$$k_s = \frac{i\xi}{1-i\xi} \tag{14}$$

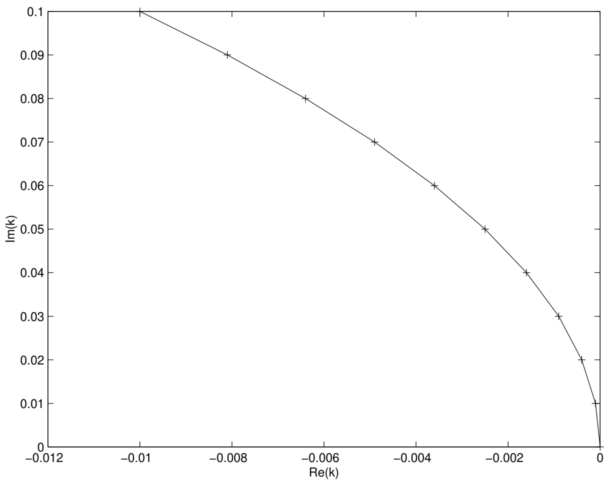

Figure 1: k_s as a function of ξ, for $0 \leq \xi \leq 0.1$. This plot uses the approximate relation (15). Note the scale on the real axis is $[-0.012, 0]$, and on the imaginary axis is $[0, 0.1]$.

To get an intuitive notion of where k_s lies in the complex plane, we may take the first-order Taylor series approximation $\frac{1}{1-i\xi} \approx 1 + i\xi$ to obtain

$$k_s \approx -\xi^2 + i\xi \tag{15}$$

Fig. 1 shows the placement of k_s as a function of ξ, for a range of small values of ξ. It is also worth observing that, for a given ξ, there is a unique critical point k_s.

Substituting Eq. (14) for k_s into expression (13) for $\partial^2\psi/\partial k^2$ leads to

$$\frac{\partial^2\psi}{\partial k^2}(k_s, \xi) = (1 - i\xi)^2, \tag{16}$$

which shows that the second k-derivative of ψ is non-zero for any real value of ξ, and so the methods of Laplace integrals may be applied.

At the next step, we study the imaginary part of ψ, in order to motivate the path of integration chosen for the Steepest Descent integral. We first define $\psi_s = \psi(k_s, \xi)$, and substitute k_s into expression (10) for ψ to obtain

$$\psi_s = i\xi + \log(1 - i\xi) \tag{17}$$

Using the general relation

$$\log(a + bi) = \log\sqrt{a^2 + b^2} + i\ \tan^{-1}\left(\frac{b}{a}\right) \tag{18}$$

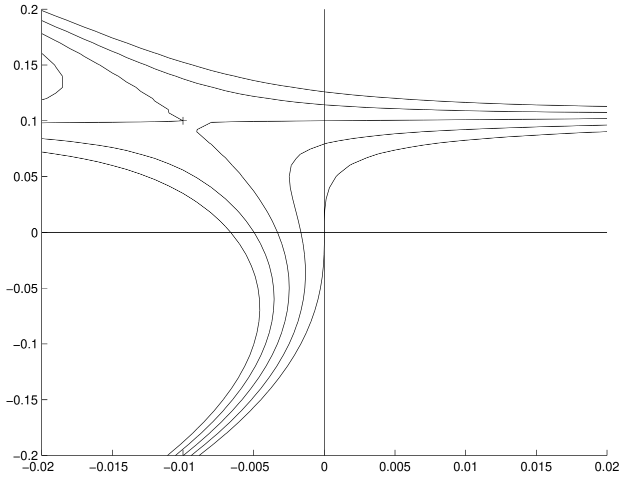

Figure 2: Several contours of $Im\ \psi$, including the contours through the origin and through the saddle point k_s. k_s is marked by the cross. Note the x-scale is $[-0.02, 0.02]$, and the y-scale is $[-0.2, 0.2]$.

we arrive at $Im\ \psi_s$ as

(19) $$Im\ \psi_s = \xi - \tan^{-1}\xi$$

From (19) we conclude $Im\ \psi_s > 0$. We also compute $Im\ \psi$ for the general case. Writing out $k = x + iy$, we have

(20) $$Im\ \psi = Im\left[(1 - i\xi)(x + iy) - \log(1 + x + iy)\right]$$

and again using the relation (18),

(21) $$Im\ \psi = -\xi x + y - \tan^{-1}\left(\frac{y}{1+x}\right).$$

This expression tells us that $Im\ \psi = 0$ for $k = 0$. And comparing with the conclusion for $Im\ \psi_s$, we note that the phases of ψ disagree between $k = 0$ and k_s. So a contour of constant $Im\ \psi$ cannot start at the origin and pass through k_s for $\xi \neq 0$.

Fig. 2 shows several contours of $Im\ \psi$, including the contours through the origin and k_s. For the path of integration, we take the contour through the origin. This path does not pass through the saddle point k_s, and hence does not produce the best approximation. However it does produce the correct leading-order term, and the integral computation is easier.

Given this path of integration, we consider the Laplace integral of Eq. (9). Denoting this integral by I, we have

(22) $$I = \int_0^\infty e^{-n\psi(k,\xi)}dk.$$

The main contribution to this integral will come from the value of the integrand at the origin. Hence, by (qualitatively) similar reasoning to that used for Laplace integrals, we replace the limits of integration with $[0, \epsilon]$. We also substitute $k = iy$. For small k, this is a valid approximation to the contour through the origin, although the true contour has the form $y \sim (3\xi x)^{1/3}$.

$$I \approx \int_0^\epsilon e^{-n\psi(iy,\xi)} i dy \tag{23}$$

We again apply the reasoning of Laplace integrals, and allow $\epsilon \to \infty$, introducing only exponentially small errors in the process. Writing out the function ψ, this leads to

$$I \approx i \int_0^\infty e^{-n[\xi y + iy - \log(1+iy)]} dy \tag{24}$$

Expanding the log term gives

$$I \approx i \int_0^\infty e^{-n\xi y} \exp\left(-n[\frac{(iy)^2}{2} - \frac{(iy)^3}{3} + ...]\right) dy. \tag{25}$$

Expanding the second exponential to first-order,

$$I \approx i \int_0^\infty e^{-n\xi y} \left(1 + \frac{ny^2}{2} - \frac{niy^3}{3} + ...\right) dy. \tag{26}$$

And recalling $P_n(n\xi) = (1/\pi)\ Re\ I$, we keep the y^3 term as the lowest-order real term to obtain

$$P_n(n\xi) \approx \frac{n}{3\pi} \int_0^\infty e^{-n\xi y} y^3 dy. \tag{27}$$

We may substitute $u = n\xi y$, and recall the definition of the gamma integral $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ to obtain

$$P_n(n\xi) \approx \frac{n}{3\pi} \frac{\Gamma(4)}{(n\xi)^4}. \tag{28}$$

and using the fact $\Gamma(m+1) = m!$ for non-negative integers m, we arrive at

$$P_n(n\xi) \approx \frac{n\left(\frac{2}{\pi}\right)}{(n\xi)^4} = \frac{n\left(\frac{2}{\pi}\right)}{x^4}. \tag{29}$$

This last equality tells us that power law tail amplitude is additive, since the transition probability (1) also had tails of size $(1/x^4)$.

The form of $P_n(n\xi)$ outside the central region, with the next-order correction included, is

$$P_n(n\xi) \approx \frac{2n}{\pi(n\xi)^4} + \frac{5!}{\pi(n\xi)^6}\left[\frac{n^2}{6} - \frac{n}{5}\right] + ... \tag{30}$$

We make this claim without proof. Eq. (30) shows that there is no leading-order Gaussian term in $P_n(n\xi)$ outside the central region.

To study P_n in the central region, we define $\zeta = x/\sqrt{n}$, and allow $n \to \infty$ in such a way that ζ stays fixed. In the process, we will also have $\xi \to 0$, and by (15), k_s will approach the origin in the complex plane.

In this scenario, we change the path of integration for P_n to pass through the saddle point, and then turn immediately right, to follow the contour $Im\ \psi = Im\ \psi_s$ to infinity, as shown in Fig. 2. On this path, the main contribution will come from the saddle point, and we may heuristically apply the Laplace integral approximation.

We first study Eq. (17) for ψ_s. With the approximation $\log(1+\epsilon) \approx \epsilon + \frac{\epsilon^2}{2}$, we make the statement

$$\psi_s \approx \frac{-(i\xi)^2}{2}. \tag{31}$$

We also note $\psi_s'' \to 1$ as $\xi \to 0$ from Eq. (16). Then applying the Laplace integration formula, we have

$$P_n(n\xi) \sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{n}}e^{-n\xi^2/2} \tag{32}$$

Writing in terms of ζ,

$$P_n(\sqrt{n}\zeta) \sim \frac{e^{-\zeta^2/2}}{\sqrt{2\pi n}} \tag{33}$$

which expresses the leading-order Gaussian behavior in the central region.