Lecture : Lévy Flights and Power-Law Tails

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1 Lévy Distributions, Walks, and Flights

A symmetric Lévy distribution is a distribution with structure function $\hat{p}(k) = \exp(-a|k|^\alpha)$. For $\alpha > 2$ the distribution has negative values in places so it is not a valid PDF. To see this note:

$$\begin{aligned}
\left\langle x^2 \right\rangle &= \int_{\mathbf{R}} x^2 p(x)\,\mathrm{d}x = -\lim_{k\to 0} \frac{\mathrm{d}^2\hat{p}}{\mathrm{d}k^2} \\
&= \lim_{k\to 0} \left[a^2\alpha^2|k|^{2(\alpha-1)} \left(\frac{\mathrm{d}|k|}{\mathrm{d}k}\right)^2 e^{-a|k|^\alpha} - a\alpha(1-\alpha)|k|^{\alpha-2}\frac{\mathrm{d}|k|}{\mathrm{d}k} e^{-a|k|^\alpha} \right. \\
&\qquad \left. - a\alpha|k|^{\alpha-1}\frac{\mathrm{d}^2|k|}{\mathrm{d}k^2} e^{-a|k|^\alpha} \right] \\
&= 0.
\end{aligned}$$

Although the derivatives of $|k|$ are not continuous at the origin, the terms in $|k|^t$ for $t > 0$ render the desired second derivative of the structure function continuous everywhere: these terms approach zero as $k \to 0$ so the limit above is justified.

When $0 < \alpha \le 2$, the inverse transform of $\hat{p}(k) = e^{-a|k|^\alpha}$ is a valid PDF and is called the (symmetric) Lévy distribution of order α: $\ell_\alpha(a, x)$. We have encountered some special cases already. The Gaussian is $\ell_2(a, x)$:

$$\hat{p}(k) = e^{-\sigma^2 k^2/2} \quad \Rightarrow \quad p(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}.$$

The Cauchy (Lorentz) distribution we saw on the homework was $\ell_1(a, x)$:

$$\hat{p}(k) = e^{-a|k|} \quad \Rightarrow \quad \ell_1(x) = \frac{a}{\pi(a^2 + x^2)}.$$

This has infinite variance and in fact $\langle |x|^m \rangle = \infty$ for all $m \ge 1$. The tails are very fat.

The Lévy distributions are attracting as we will see in lecture 20. They are clearly stable:

$$\hat{p}_n(k) = [\hat{p}(k)]^n = e^{-an|k|^\alpha}.$$

This is clearly the same form, but with different constants. In fact, $a_n = na$. The constant a is additive much like the variance. It is, in some sense, a generalized cumulant.

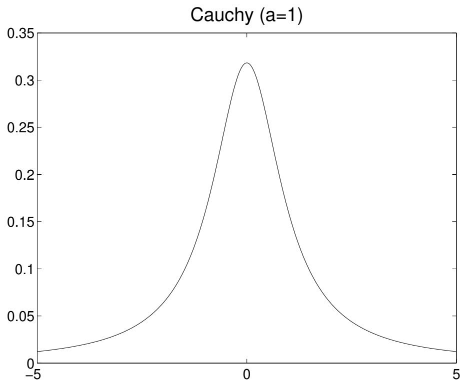
When each step takes the same time regardless of length, the random "walk" is called a flight rather than a walk. The term random walk is reserved for when each step takes time proportional to it's length. Flights with Lévy-distributed steps are called Lévy flights. If one uses a walker that can only move at constant speeds the scaling changes drastically and we have, instead, Lévy walks. It has been claimed that the wandering length of an albatross from it's nest can be modeled with Lévy walks.

The exact solution for a (symmetric) Lévy flight with these step sizes is

$$P_n(x) = \ell_\alpha\left(a, \frac{x}{n^{1/\alpha}}\right)\frac{1}{n^{1/\alpha}}.$$

They all have roughly the same shape for different α, but x scales as $t^{1/\alpha}$ (or $n^{1/\alpha}$). For $\alpha < 2$ the width of the distribution goes as $1/\alpha$ so we have super diffusion. The width of the distribution cannot be measured by standard deviations, but we can just as easily pick other measures of width such as percentiles.

What do the Levy distributions look like? The look similar to a Gaussian in the center, but the tails are much flatter and contain a significant portion of the probability mass. Here is a plot of the Cauchy distribution ($\alpha = 1$):



In the central region they look something like a normal distribution, but the asymptotic distribution in the tails is

$$\ell_\alpha(x) \sim \frac{a\,\sin(a\pi/2)\,\Gamma(\alpha)\alpha}{\pi\,|x|^{1+\alpha}} = \frac{A}{|x|^{1+\alpha}} \quad \text{as} \quad x \to \infty.$$

for $0 < \alpha < 2$. This shows we get a power-law tail for $0 < \alpha < 2$. (The expansion fails for $\alpha = 2$.) Since $a_n = na$, $A_n = nA$ so the tail amplitudes are additive for Lévy flights. This is actually always true for any power law tail. We now give an argument for this.

2 Power Law Tails are Additive

Let $\log \hat{p}(k) = f_a(k) + f_s(k)$ where $f_a(k)$ is analytic and $f_s(k) \sim -a|k|^\alpha$ as $k \to 0$ with a real and finite. Then

$$P_n(x) \sim \frac{A_n}{|x|^{1+\alpha}} \quad \text{as} \quad x \to \infty$$

where $A_n = nA$ and

$$A = \frac{a \sin(\alpha\pi/2)\Gamma(\alpha)\alpha}{\pi}.$$

"Proof": Note that $\log \hat{p}(k) \sim -c_2 k^2/2 - a|k|^\alpha$ where $c_2 = f_a''(0)$. This is an even function. As we are primarily concerned with small k (i.e. large x), these terms will dominate. Thus we consider structure functions with $\log \hat{p}(k)$ even. It should not change the result.

$$P_n(x) = \int_{\mathbf{R}} e^{ikx}[\hat{p}(k)]^n \frac{dk}{2\pi} = \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{ikx}[\hat{p}(k)]^n \, dk$$

because $\hat{p}(k)$ is even. Rewrite the exponent of the integral as $-x\,\psi(k,x,n)$ so that

$$P_n(x) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty \exp\left\{-x\,\psi(k,x,n)\right\}.$$

Let $f_a(k) = F_a(ik)$ and $f_s(k) = F_s(|k|)$. Note that $F_a(x)$ is a real valued function on $\mathbf{R}$ as is $F_s(x)$. We have

$$\psi(k,x,n) = -ik - \frac{n}{x}\left[F_a(ik) + F_s(|k|)\right].$$

For $\alpha > 1$, $|k|^\alpha$ decays more quickly than ik so $\psi \sim -ik$ as $k \to 0$, and $\operatorname{Im}(\psi) \sim -\operatorname{Re}(k)$. Thus the steepest descent contour approaches the origin along the positive imaginary axis. If $\alpha = \infty$ (i.e. no singularity) then there is no contribution at the endpoint of integration at $k = 0$ and the imaginary axis is a steepest descent contour which gives no contribution anywhere. If $f_s \neq 0$ and $1 < \alpha < \infty$, f_s will lead to a real contribution at $k = 0$. Make the substitution $k = iy$ for $x > 0$:

$$P_n(x) \sim -\frac{1}{\pi} \operatorname{Re} \int_0^\epsilon e^{-xy} e^{nF_a(-y)} e^{nF_s(iy)} \, i\, dy \quad \text{as} \quad |x| \to \infty.$$

The second exponential in the integral is real and goes to 1 as $k \to 0$ so there is no contribution to leading order. $e^{nF_s(iy)} \sim 1 - a_n(iy)^\alpha$. The term $-a_n(iy)^\alpha$ determines leading order behavior. Plugging this into our expression for $P_n(x)$ yields

$$P_n(x) \sim -\frac{1}{\pi} \operatorname{Im} \int_0^\epsilon e^{-xy}\left[-a_n(iy)^\alpha\right] dy.$$

We have absorbed the i in $d(iy)$ into the Re operator. As $x \to \infty$ and we assume the functions decay as necessary, then we can pick x large enough so that ϵ can be taken as large as we want. Thus

$$P_n(x) \sim \frac{a_n}{\pi} \operatorname{Im}(i^\alpha) \int_0^\infty e^{-xy} y^\alpha \, dy = \frac{a_n}{\pi} \sin\left(\frac{a_n \pi}{2}\right) \int_0^\infty e^{-u}\left(\frac{u}{x}\right)^\alpha \frac{du}{x}.$$

The last integral is simple and yields $\Gamma(\alpha+1)/x^{1+\alpha} = \alpha\Gamma(\alpha)/x^{1+\alpha}$. Similar calculations for $x < 0$ may be carried out. We get the result

$$P_n(x) \sim \frac{a_n \sin(a_n\pi/2)\Gamma(\alpha)\alpha}{|x|^{1+\alpha}\pi} \quad \text{as} |x| \to \infty.$$

The case when $\alpha = 1$ is slightly different, but the same analysis works. In this case $\psi \sim -\left(i + \frac{n}{x}\right) k$ as $k \to 0$. The steepest direction from the origin is now

$$k \sim \frac{y}{-(i + n/x)} = \frac{iy}{1 - i(n/x)} \sim iy \quad \text{as} \quad \frac{x}{n} \to \infty.$$

Thus the analysis also works for $\alpha = 1$. It is consistent with the exact result:

$$P_n(x) = \frac{a_n}{\pi(a_n^2 + x^2)} \sim \frac{(a/\pi)}{|x|^2} \quad \text{as} \quad |x| \to \infty.$$

Next time we will continue with the analysis for $\alpha < 1$.