Lecture 19: Lévy Flights and Extreme Events

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1 Additivity of Power-Law Tail Amplitude (continued)

Adopt the notation from last lecture. We have

$$\log \hat{p}(k) \sim \underbrace{\sum_{m=2}^{\infty} \frac{c_m (ik)^m}{m!}}_{f_a(k)} - \underbrace{a|k|^\alpha}_{f_s(k)}.$$

Here $f_a(k) = F_a(ik)$ where $F_a(x)$ is real on $\mathbf{R}$ and goes to zero in the right half plane. $F_s(|k|) = f_s(k)$ with $F_s(x)$ real on $\mathbf{R}$, analytic, and going to zero in the right half plane. We have already dealt with $\alpha \geq 1$ so now we consider $\alpha < 1$. We have

$$P_n(x) = \frac{1}{\pi} \Re \int_0^\infty e^{ikx - ak^\alpha} \, \mathrm{d}k.$$

Let $\psi = ikx + ak^\alpha$ so

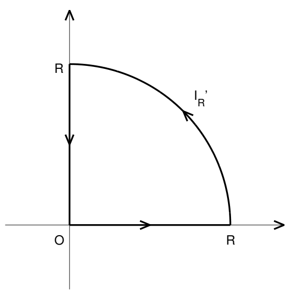
$$\frac{\partial \psi}{\partial k} = -ix + a\alpha k^{\alpha-1} \quad \Rightarrow \quad k_s = \left(\frac{ix}{a\alpha}\right)^{1/(\alpha-1)}.$$

For $\alpha > 1$, $k_s \to \infty$ as $x \to \infty$ but for $\alpha < 1$, $k_s \to 0$ as $x \to \infty$. Thus the saddle point does not work well. We must use a different strategy.

Let

$$I = \int_0^\infty e^{ikx} e^{nF_a(ik)} e^{nF_s(k)} \, \mathrm{d}k$$

so that $P_n(x) = \Re(I)/\pi$. Assuming that the asymptotics of $\log \hat{p}(k)$ hold everywhere (and they do for Lévy flights), we pick the following contour (and assume there are no singularities inside):



On I'_R we have $z = Re^{i\theta}$ so

$$\begin{aligned}
|I'_R| &= \left|\int_0^{\pi/2} \exp\left[ixRe^{i\theta}\right] \exp\left[nF_a(iRe^{i\theta})\right] \exp\left[nF_s(Re^{i\theta})\right] Re^{i\theta}\,\mathrm{d}\theta\right| \\
&\le \int_0^{\pi/2} \left|\exp\left[ixRe^{i\theta}\right] \exp\left[nF_a(iRe^{i\theta})\right] \exp\left[nF_s(Re^{i\theta})\right] Re^{i\theta}\right| \mathrm{d}\theta \\
&= \int_0^{\pi/2} \exp\left[-xR\sin\theta\right] M_R^n M'_R R\,\mathrm{d}\theta
\end{aligned}$$

where we assume

$$\exp\left[F_a(Re^{i\theta})\right] < M_R \quad \text{and} \quad \left|\exp\left[nF_s(Re^{i\theta})\right]\right| < M'_R \quad \text{for} \quad 0 \le \theta \le \pi/2$$

with $M_R \to 0$ as $R \to \infty$. This yields

$$|I'_R| \le RM_R^n M'_R \int_0^{\pi/2} \exp\left[-xR\sin\theta\right] \le RM_R^n M'_R(\pi/2)$$

by Jordan's Lemma. This goes to 0 as $R \to \infty$. Thus I is equal to the integral up along the imaginary axis which yields

$$P_n(x) = \frac{1}{\pi}\Re \int_0^\infty e^{-xy} e^{nF_a(-y)} e^{nF_s(iy)} i\,\mathrm{d}y$$

which is the same as last time so we get the desired result:

$$P_n(x) \sim \frac{a_n \sin\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)\alpha}{\pi|x|^{1+\alpha}}$$

as $x \to \infty$. Unfortunately, along the way we have reduced the generality of application a great deal. Nevertheless the results still apply to Lévy flights. We can go even farther with the relatively simple Lévy distributions, as we now see.

2 Expansion of Symmetric Lévy Laws

2.1 Large x expansion

The large x asymptotic expansion for $\ell_\alpha(a,x)$ (with structure function $\hat{\ell}_\alpha(a,k) = e^{-a|k||alpha}$) is

$$\ell_\alpha(a,x) \sim \frac{1}{\pi x} \sum_{m=1}^{\infty} \frac{a^m \sin(\alpha m\pi/2)\Gamma(\alpha m+1)(-1)^{m-1}}{m!\,|x|^{\alpha m}}$$

as $x \to \infty$. The leading order term is the familiar expression

$$\ell_\alpha(a,x) \sim \frac{a_n \sin\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha)\alpha}{\pi|x|^{1+\alpha}}.$$

Note that this series is divergent for $\alpha > 1$ and convergent for $\alpha \le 1$ where $|x|^\alpha > a$. This is a Laurent series.

2.2 Small x expansion

For small x we have

$$\ell_\alpha(a,x) = \frac{1}{\pi}\int_0^\infty \cos(kx)e^{-ak^\alpha}\,\mathrm{d}k \sim \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\int_0^\infty (kx)^{2m}e^{-\alpha k^\alpha}\,\mathrm{d}k$$

from expanding $\cos(kx)$. This yields

$$\ell_\alpha(a,x) \sim \frac{1}{\pi\alpha a^{1/\alpha}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\Gamma\left(\frac{2m+1}{\alpha}\right)\left(\frac{x}{a^{1/\alpha}}\right)^{2m}.$$

This series diverges for $\alpha < 1$ and converges for $1 \leq \alpha < 2$ for $|x|^\alpha < a$. To leading order it is

$$\ell_\alpha(a,x) \sim \frac{\Gamma(1/\alpha)}{\pi\alpha a^{1/\alpha}}.$$

Thus the center gets sharper and higher as $\alpha \to 0$. Montroll-Bender tell us

$$\ell_\alpha(a,x) \sim \frac{\Gamma(1/\alpha)}{\pi\alpha}e^{-x^2/2\sigma^2}$$

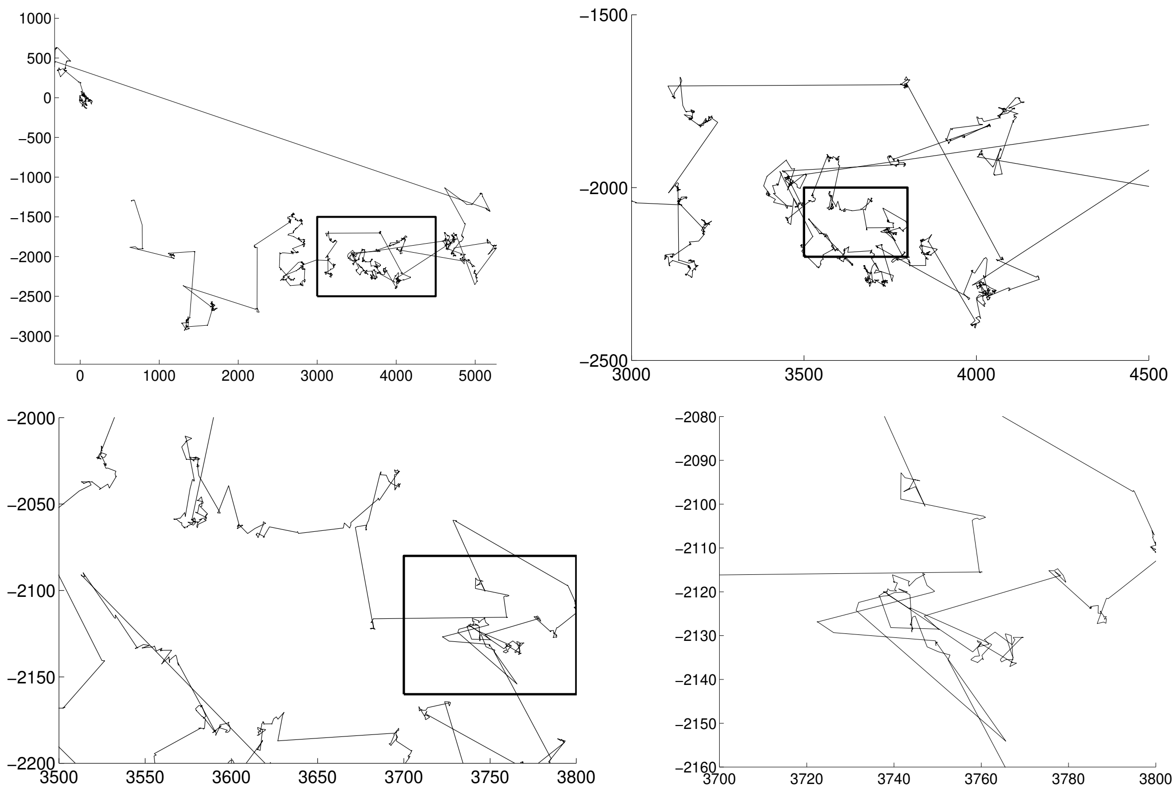
where

$$\sigma = \sqrt{\frac{\pi}{2}}\frac{(1-1/e)\alpha}{\sqrt{1/\alpha}}.$$

This goes to the δ function as $\alpha \to 0$.

2.3 A Typical Lévy trajectory

Since $\sigma^2 = \infty$ for Lévy flights, we have lost a length scale. Thus a Lévy flight looks the same at all length scales. Here is an example. The pictures are shown in order of increasing magnification, but they all look qualitatively the same.



3 Extreme Events and Moments

Given random iid displacements ΔX_i, what is the distribution of the largest steps $\Delta X(n)$. In extremes, only tails are really relevant, at least for large n. For different tail decays, we have different distributions.

3.1 Bernoulli: bounded tail

The largest is also bounded. These produce a Weibull distribution. This is important in the theory of fractures.

3.2 Exponential decay

Assume CDF is approximated by $P(\Delta x < x) \sim 1 - e^{-x/x_0}$. Then $P(\Delta X_{(n)} < x) < X) = [P(\Delta X, x)]^n$ exactly. Far in the tail this becomes

$$P(\Delta X_{(n)} < x) \sim \left(1 - e^{-x/x_0}\right)^n = \left(1 - \frac{\exp\left(-\frac{x - x_0 \log n}{x_0}\right)}{n}\right)^n .$$

Note that $\lim_{n\to\infty}(1-z/n)^n \to e^z$. Thus if we let

$$Z_n = \frac{\Delta X_{(n)} - x_0 \log n}{x_0}$$

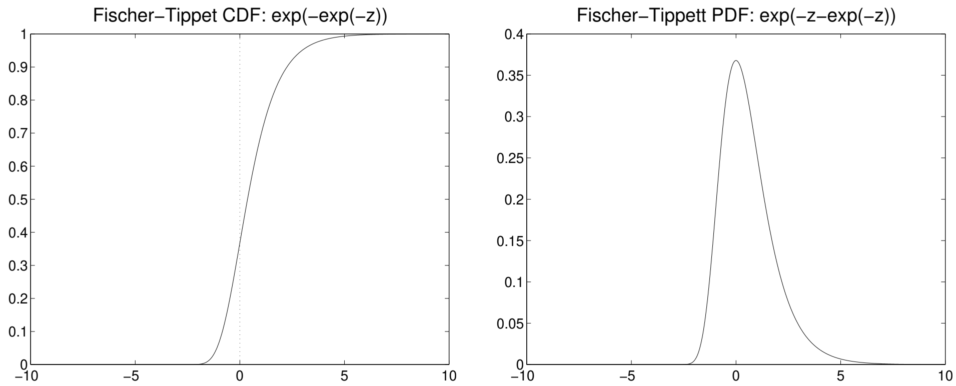
and take the large n limit we get

$$P(Z_n \le z) = e^{-e^{-z}}.$$

This is a Fischer-Tippet distribution (although it is sometimes called a "Gumbel" distribution). This implies that to lowest order we have

$$\begin{aligned}\left\langle \Delta X_{(n)} \right\rangle &\sim x_0 \log n + \gamma + \cdots \\ \sigma^2_{\Delta X_{(n)}} &\sim x_0^2 + \frac{\pi^2}{6} + \cdots\end{aligned}$$

where γ is Euler's constant. The CDF and PDF of this distribution are shown below:



If ΔX_i are normal, then $\left\langle \Delta X_{(n)} \right\rangle$ scales approximately as $\sqrt{\log n}$. Note that to get into the tail, the walker must make $O(n)$ steps in the same direction, and this is extremely rare.

3.3 Power-Law tails

In the tails (large $|x|$), $p(x) \sim A/|x|^{1+\alpha}$. In this case $\left\langle \Delta X_{(n)} \right\rangle$ scales approximately as $n^{1/\alpha}$. (The distribution is actually a Fréchet distribution.) This scaling has implications for the sum

$$\sum_{i=1}^{n} \Delta X_i^m.$$

For $\alpha \le m$, a single term can dominate the sum and all moments $\ge \alpha$ diverge. However, with exponential decay of the structure function, a single term is only of order $\log n$, so a well-defined mth moment will exist for *all* m.