Lecture 20: Levy stable laws and renomarlization

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1 Extreme Events with power-law tails

Suppose

$$p(x) = \frac{A}{x^{1+\alpha}} \text{as } x \to \infty \text{(PDF)}$$

$$P(x) = 1 - \frac{A}{\alpha x^{\alpha}} \text{as } x \to \infty \text{(CDF)}$$

As before, if Δx_i are i.i.d. steps,

$$\begin{aligned} Prob(\Delta x_{(n)} < x) &= P(x)^n \\ &= \left(1 - \frac{A}{\alpha x^{\alpha}}\right)^n \text{ as } n \to \infty \\ &= \left(1 - \frac{\left(\frac{(An/\alpha)^{1/\alpha}}{x}\right)^{\alpha}}{n}\right)^n \end{aligned}$$

where $\Delta x_{(n)}$ is the biggest step so far.
If we let

$$Z_n = \frac{\Delta x_{(n)}}{\left(\frac{A}{\alpha} \cdot n\right)^{1/\alpha}}$$

one can show that the probability is reduced into the following form :

$$Prob(Z_n < z) = e^{-Z^{-\alpha}}$$

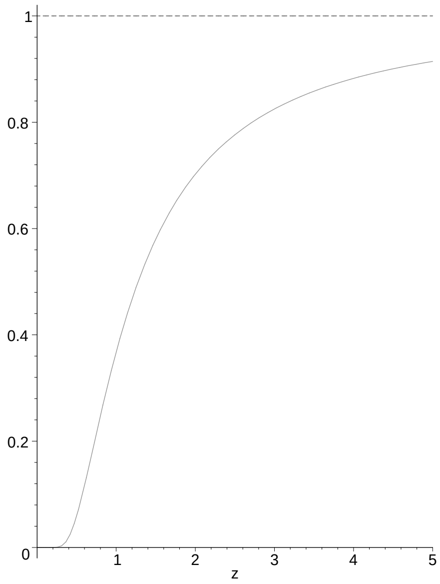

Figure 1: CDF $\alpha = 3/2$

as $n \to \infty$ with z fixed.

The shape of the distribution is as follows. In large Z limit, $Prob(Z_n < z) = 1 - 1/Z^{\alpha}$ and it approaches to unity. And, at $z \to 0$, this CDF has essential singularity and its slope is near zero.

PDF is skewed around 1. In high z, PDF is asymptotically decreasing with power law as $1/z^{1+\alpha}$. This distribution is known as Frechet Distribution (1926).

The mean and the standard distribution are as follows.

$$< \Delta x_{(n)} >= \left[\left(\frac{A}{\alpha} \right) \cdot n \right]^{1/\alpha}$$

$$\sigma_{\Delta x_{(n)}} = n^{1/\alpha}$$

Note both average and standard deviation have the same scaling.

Consider the sum

$$\frac{1}{n} \sum_{i=1}^{n} \Delta x_i^n$$

When $m > \alpha$, then the largest term in the sum (scales faster than n) dominates the sum. Largest term $\frac{n^{m/\alpha}}{n}$ grows and hence a finite value for the moment cannot exist.

2 Weight of the tail of a Levy Flight

One example of Levy flight is Cauchy distribution, which is the case with $\alpha = 1$

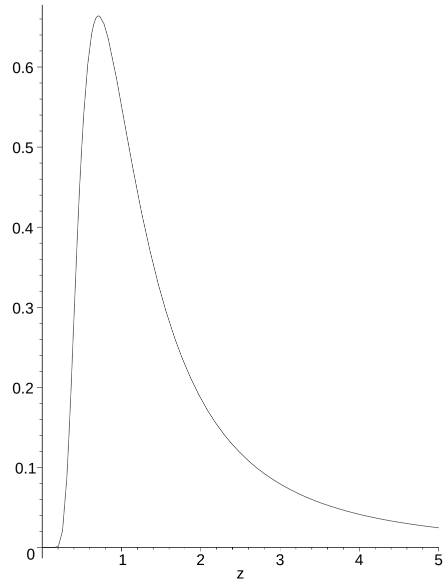

Figure 2: PDF $\alpha = 3/2$

$$p(x) = \frac{1}{\pi(1+x^2)}$$

$$P(x) = \frac{1}{\pi} tan^{-1}(x) + \frac{1}{2}$$

$$\alpha = 1$$

$$\alpha \to 0$$

This distribution has Gaussian center.

$$l_\alpha(x) = \frac{\Gamma(1/\alpha)}{\pi\alpha} e^{-x^2/2\sigma_{\alpha^2}}$$

$$\sigma_\alpha = \sqrt{\frac{\pi}{2}} \frac{\left(1 - \frac{1}{e}\right)^\alpha}{\Gamma(1/\alpha)}$$
$$= (0.40)\Phi(x/\sigma_\alpha)$$

This means the central region explains about 40

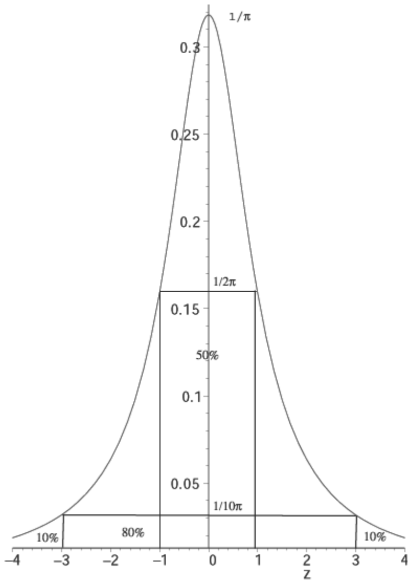

Figure 3: Cauchy distribution

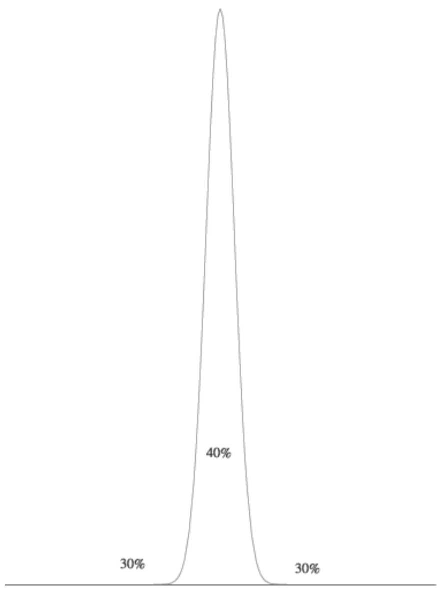

Figure 4: $\alpha \to 0$

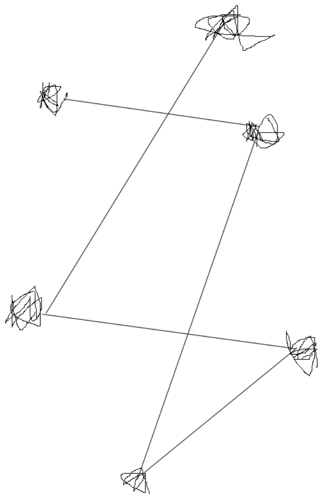

Figure 5: $\alpha \to 0$, Levy flight steps

3 Some applications

3.1 Financial Time Series

Mandlbrot considered Levy flight and Bouchaud and Potetrs considered Truncated Levy flight. Since avaliable data are finite, real situation can be approximated with truncated Levy flight.

Truncated Levy flight (TLF) : If $< \Delta x_{(n)} >= \left(\frac{A}{\alpha} \cdot n\right)^{1/\alpha}$, then for $\left(\frac{A}{\alpha} \cdot n\right)^{1/\alpha} << x$, $n << \frac{\alpha}{A} \cdot x_1^{\alpha} (== n)$, the flight looks like Levy, but for for $n >> n_1$, then Gaussian distribution is obtained (CLT holds).

3.2 Polymer Surface Adsorption

The adsorption of polymer on the surface has number of contact points. The points of contact can be modeled as self-avoiding Levy flight.

This has been analyzed by Bouchaud and George (Physics reports, 195, 127 (1998)). This system has interesting phase diagram.

4 Renormalization and stable laws

What are the possible limiting distributions of sum of random variables (iid)?
mean = 0

$$\hat{P}_n(k) = \hat{P}_1(k)^n$$

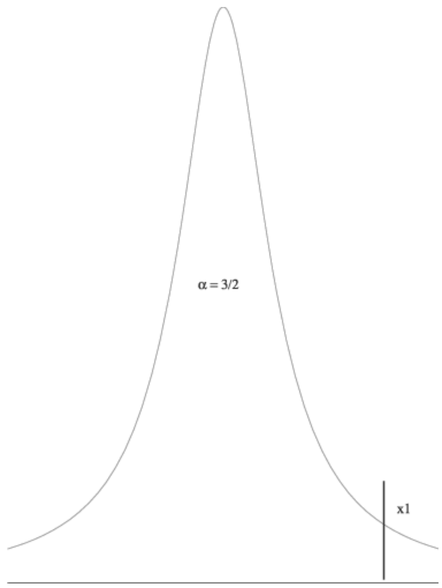

Figure 6: $\alpha = 3/2$

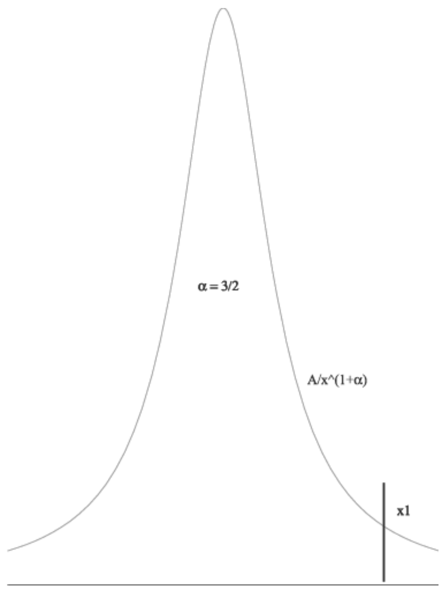

Figure 7: $\alpha \to 0$

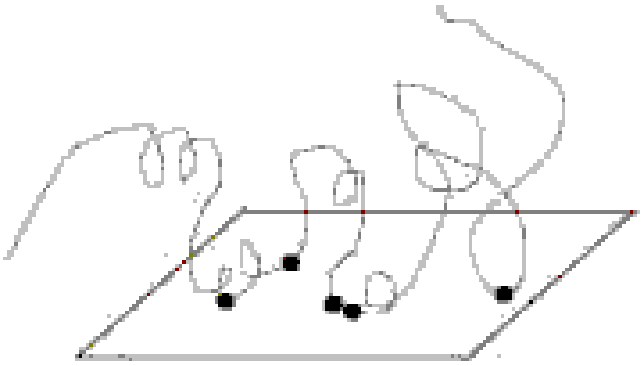

Figure 8: Polymer surface adsorption

Look for fixed points of this recursion.
Suppose $Z_n = \frac{X_n}{a_n}$ is the appropriately rescaled random variable, which is stable.

$$F_n(z) = \text{PDF of } Z_n,\ \hat{F}_n(k) = F.T.$$

For independent steps,

$$\hat{F}_n(a_n k) = \hat{F}_1(k)^k$$

If there is a correlation time n_c, then for $m >> n_c$, we have

$$\hat{F}_{mn}(a_{mn}k) = \hat{F}_m(a_m k)^n$$

(Take n super steps)

$$\hat{F}_{mn}\left(\frac{a_{mn}}{a_m}k\right) = \hat{F}_m(k)^n$$

This is a renormalization group.

$$\hat{F}_m \to R_n \hat{F}_m = \hat{F}_{mn}$$

$$R_{mn} = R_m R_n$$

(semi group) To get a fixed point,

$$\hat{F}_m \to \hat{F}(k)$$

then we need : scaling :

$$\frac{a_{mn}}{a_m} \to C_n$$

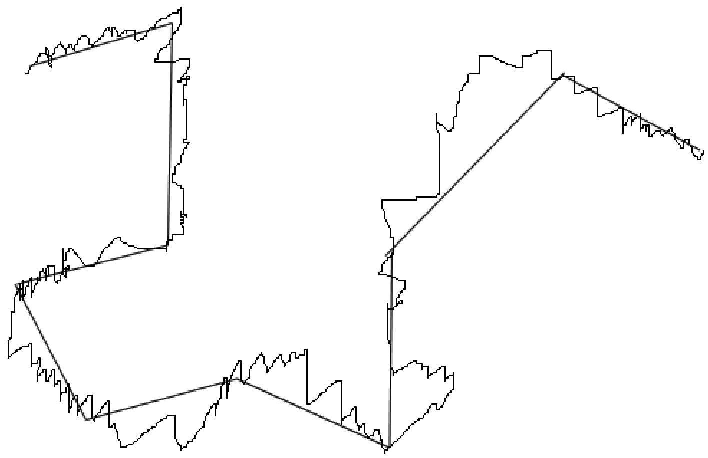

Figure 9: Renormalization

as

$$m \to \infty$$

Let

$$a_n = n^{1/\alpha} L_n$$

($1/\alpha$ is $\alpha > 0$ power)
stability condition :

$$\frac{a_{mn}}{a_m} = \frac{(mn)^{1/\alpha}}{m^{1/\alpha}} \frac{L_{mn}}{L_m} \to n^{1/\alpha} = C_n$$

if $\frac{L_{mn}}{L_m} \to 1$, ("slowly varying")
ex) $L_n = (logn)^{\mu}$
Shape of the limiting distribution must satisfy

$$\hat{F}(n^{1/\alpha} k) = \hat{F}(k)^n$$

for all $n \epsilon N$
Note : Only solution

$$\hat{F}(k) = e^{-\nu(signk)|k|^{\alpha}}$$

Note $\hat{F}(0) = 1$
If p(x) = even = p(-x), then $\hat{F}(k) = \hat{F}(-k)$ even. Therefore ν(sign k) = a = constant

$$\hat{F}(k) = e^{-a|k|^{\alpha}}$$

Symmetric Levy distribution $l_{\alpha}(x)$

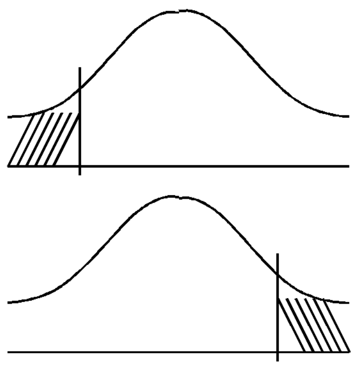

Figure 10: No caption

More generally,

$$\hat{F}(-k) = \hat{F}(k)^*$$

$$\rightarrow \nu(sgnk) = C_1 + iC_2 sgn(k)$$

One can show :

$$\hat{F} = e^{-a|k|^\alpha (1 - i\beta tan(\frac{\alpha\pi}{2}) sgn(k))}$$

If $\beta \neq 0$ and $\alpha = 1$, this $= -\frac{2}{\pi} log|k|$
$-1 <= \beta <= 1$ asymmetry parameter.

$$\rightarrow l_{\alpha,\beta}(x)$$

5 Basins of Attraction

(Figure 11)
Theorem (Gnendenko, Doeblin) p(x) is in the basin of attraction of $l_{\alpha,\beta}(x)$ iff the CDF $p(x) = \int_{-\infty}^{x} p(x)dx$ satisfies
(i)

$$limit_{x\rightarrow\infty} \frac{p(-x)}{1 - p(x)} = \frac{1-\beta}{1+\beta}$$

(ii) For any $r > 0$,

$$limit_{x\rightarrow\infty} \frac{1 - p(x) + p(-x)}{1 - p(rx) + p(-rx)} = r^\alpha$$

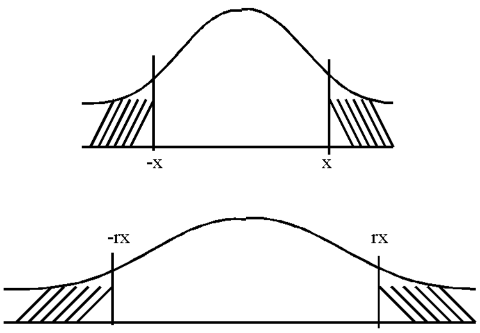

Figure 11: No caption

Theorem (Gnedenko) p(x) is basin of attraction of $l_{\alpha,\beta}(x)$ with $a_n = n^{1/\alpha}$ iff

$$p(x) = \begin{cases} A_+/x^{1+\alpha} & : \quad x \to +\infty \\ A_-/|x|^{1+\alpha} & : \quad x \to -\infty \end{cases}$$

where

$$\beta = \frac{A_+ - A_-}{A_+ + A_-}$$