Lecture 24: Brownian Motion in Fluid Mechanics (by Michael Brenner)

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This guest lecture was presented by Michael Brenner of the Department of Mathematics. Two main topics were discussed which incorporate Brownian motion into fluid mechanics: Taylor dispersion and the relaxation of particles in a potential field. The former topic was discussed in much greater detail.

1 Taylor Dispersion

The canonical Taylor dispersion problem [1] is unidirectional Poiseuille (parabolic) flow in a cylindrical tube of radius R. In the case where no flow exists, a slug of solute will spread with the scaling $\sqrt{2Dt}$, with D the molecular diffusivity. In the case where there is flow, but no diffusion, then a uniform slug of solute will spread parabolically. For intermediate cases, where there exists both diffusion and flow, Taylor derived the following conservation equation for the cross-sectional average concentration, $\langle c \rangle$,

$$\partial_t \langle c \rangle + \langle U \rangle \partial_z \langle c \rangle = D^* \partial_{zz} \langle c \rangle , \tag{1}$$

where $\langle U \rangle$ is the average velocity in the (z) direction of flow, and D^* is the Taylor dispersivity,

$$D^* = D + \frac{1}{48} \frac{\langle U \rangle^2 R^2}{D}. \tag{2}$$

Rather than derive this result, we will instead focus on the dispersion between two flat plates, separated by a distance $2H$, a problem first solved by Woodling in 1960 [2]. With a coordinate system based upon the centerline, where x is the direction of flow and y is the transverse direction, the velocity profile for unidirectional pressure-driven flow in the x-direction is

$$U(y) = \frac{U_0}{H^2} \left(H^2 - y^2 \right) . \tag{3}$$

In the preceding, U_0 is a characteristic fluid velocity.

The evolution of the solute concentration is given by the convection-diffusion equation,

$$\partial_t c + U\partial_x c = D\nabla^2 c. \tag{4}$$

Define the average solute concentration,

$$\langle c\rangle = \frac{1}{2H}\int_{-H}^{H} c(x,y;t)dy, \tag{5}$$

as well as the average fluid velocity,

$$\langle U\rangle = \frac{1}{2H}\int_{-H}^{H} U(y)dy = \frac{2}{3}U_0. \tag{6}$$

Express the concentration field and the velocity field as perturbations about their average values,[1]

$$c = \langle c\rangle + c'(x,y;t), \tag{7}$$

$$U = \langle U\rangle + U'(y). \tag{8}$$

We already know the value of the perturbation to the velocity field,

$$U' = U_0\left[\frac{1}{3} - \left(\frac{y}{H}\right)^2\right], \tag{9}$$

but will need to solve for c'.

Taylor dispersion is based upon the assumption that diffusion across the tube cross-section is fast compared to the time for convection,

$$\frac{H^2}{D} \ll \frac{L}{U_0}. \tag{10}$$

As a corollary, the perturbations to the concentration field are small.

Introduce the expressions in (7) and (8) into the conservation equation (4),

$$\partial_t\langle c\rangle + \partial_t c' + [\langle U\rangle + U']\,\partial_x[\langle c\rangle + c'] = D\nabla^2[\langle c\rangle + c']. \tag{11}$$

Take the average of the latter equation, noting that (i) the average quantities are unaltered and (ii) the average of a primed quantity vanishes,

$$\partial_t\langle c\rangle + \langle U\rangle\,\partial_x\langle c\rangle + \langle U'\partial_x c'\rangle = D\partial_{xx}\langle c\rangle. \tag{12}$$

In order to arrive at an equation similar to (1), we must evaluate the fluctuation term $\langle U'\partial_x c'\rangle$. Subtract (12) from (11),

$$\partial_t c' + U'\partial_x\langle c\rangle + \langle U\rangle\,\partial_x c' + U'\partial_x c' - \langle U'\partial_x c'\rangle = D\nabla^2 c'. \tag{13}$$

[1] Although we use the expression perturbation, the expressions in eqs. (7) and (8) are exact. The concept of a perturbation lies in the assumption that c' is small as a result of the rapid transverse diffusion.

Using the assumption that the diffusion is fast, $\partial_t c' \to 0$. Moreover, the assumption that c' is small allows one to neglect the terms $\partial_x c'$ and $\partial_{xx} c'$, but not $\partial_{yy} c'$ since the gradients in the direction of the walls may be large to satisfy the no flux condition prevailing there. Accordingly, the following relationship must be satisfied,

$$U' \partial_x \langle c \rangle = D \partial_{yy} c'. \tag{14}$$

It should be noted that Taylor's analysis begins with the latter equation. Integrate twice, applying the no-flux boundary condition $\partial_y c' = 0$ at $y = \pm H$ to arrive at the expression for c',

$$c' = \partial_x \langle c \rangle \frac{U_0}{3D} \left[\frac{y^2}{2} - \frac{y^4}{4H^2} \right] + K(x,t), \tag{15}$$

where $K(x,t)$ is a constant of integration. Since $\langle c' \rangle = 0$, the unknown constant has the numerical value[2]

$$K = \partial_x \langle c \rangle \frac{U_0}{3D} \left(-\frac{7}{60} H^2 \right). \tag{16}$$

Take the x-derivative of the equation for c',

$$\partial_x c' = \partial_{xx} \langle c \rangle \frac{U_0}{3D} \left[\frac{y^2}{2} - \frac{y^4}{4H^2} - \frac{7}{60} H^2 \right]. \tag{17}$$

Multiply by U' and take the average to arrive at the result

$$\langle U' \partial_x c' \rangle = \partial_{xx} \langle c \rangle \frac{U_0^2 H^2}{D} \left(\frac{-8}{945} \right). \tag{18}$$

Substitute the latter expression into (12) to arrive at the coarse-grained expression (1) for the concentration. The Taylor dispersivity (2) in the flat plate case is

$$D^* = D + \frac{2}{105} \frac{\langle U \rangle^2 R^2}{D}, \tag{19}$$

where we have made use of the relationship between U_0 and $\langle U \rangle$ (6).

2 Motion in a Potential Field

This short subsection of the lecture is based upon a paper by Meiners and Quake [3]. Consider first the hydrodynamics of a single particle in solution. Its diffusivity is given by the Einstein relationship,

$$D = MkT, \tag{20}$$

[2] In lecture, it was stated that the constant K is irrelevant to the averaging. Indeed, the final integration is independent of K. However, for completeness (and accurate derivations of Taylor dispersivity in more complicated systems), it is useful to note that there does exists a numerical value for K which satisfies the conservation of mass.

where M is the particle mobility, k is Boltzmann's constant, and T is the temperature. For a spherical particle of radius a in a fluid of viscosity η, its mobility is the inverse of the Stokes resistance

$$M^{-1} = 6\pi\eta a. \tag{21}$$

However, the system is more complicated for two particles. The diffusivity is still given by (20), but the mobility is no longer a constant scalar. Rather, the hydrodynamic mobility is a tensor, and the hydrodynamic effects decay on the order of $1/r$. More detailed information on particle mobility in low Reynolds number flow is available elsewhere [4].

The question addressed in the paper is the correlation between the fluctuations of two spheres when they are pinned in place by optical tweezers. Meiners and Quake observed that, contrary to intuition, that the sphere motion tends to be anticorrelated. This occurs due to the hydrodynamic interactions which slow the relaxation process when the spheres move in opposite directions. A full explanation of this phenomena is available in the reference [3].

References

[1] G.I. Taylor, Proc. R. Soc. London, Ser A **219**, 186 (1953).

[2] R.A. Woodling, J. Fluid Mech. **7**, 501 (1960).

[3] J.C. Meiners and S.R. Quake, Phys. Rev. Lett. **82**, 2211 (1999).

[4] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Kluwer, Netherlands, 1983.