Intermediate Asymptotics of Random Flights III

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March 6, 2001

1 Random Walks Versus Random Flights

At this point of the course it is worth making a distinction between random walks and random flights. Both terms have been used quite liberally so far because they have been equivalent in the cases we considered, but the difference will come to play a role. A random flight corresponds to a process in which the position of our "bird" is given at fixed time intervals. No matter how long is a step, it is always considered to be taken in the same amount of time. For a random walk though, the time between two steps will be directly proportionnal to the length of the step, as if the walker had constant velocity. This difference becomes important when we consider distribution with fat tails in which steps can be very large and thus can make the distinction matter.

2 Intermediate Asymptotics

2.1 Dimensionless Formulation with an Artificial Time Scale

This being said, let us return to the derivation of intermediate asymptotics. We restrict ourselves to random flights and consider a different method to solve the equation given by the Kramer-Moyal expansion. The first thing we will do here is to pick an arbitrary time scale T with the restriction that $T \gg \tau$, where τ is the time it takes to make one step. The advantage of this method is that our time scale is now constant, and this will make taking time derivatives a lot easier.

We now non-dimensionalize the variables using this time scale:

$$\begin{aligned}
\tilde{t} &= \frac{t}{T} \\
L &= \sqrt{2DT} \\
\tilde{x} &= \frac{x}{L} = \frac{x}{2DT} \\
\tilde{C} &= \frac{C}{C_0}
\end{aligned}$$

Here the initial concentration is taken to be $C_0 = M_0/\sqrt{2DT}$ where M_0 is the initial mass of random flyers. In our case, $M_0 = 1$ since we are interested in only one flyer. We can now define a small parameter

$$\epsilon = \frac{\tau}{T}$$

and this is now a constant (contrasting with our previous small parameter which was time dependent).

Let us recall the Kramer-Moyal expansion:

$$\frac{\partial C}{\partial t} - D\frac{\partial^2 C}{\partial x^2} = -\tau^{1/2}\mathcal{D}_{\ni}\frac{\partial^{\ni}\mathcal{C}}{\partial \S^{\ni}} + \tau\left(\mathcal{D}_{\triangle}\frac{\partial^{\triangle}\mathcal{C}}{\partial \S^{\triangle}} - \frac{\infty}{\in}\frac{\partial^{\in}\mathcal{C}}{\partial \sqcup^{\in}}\right) + \mathcal{O}(\tau^{\ni/\in})$$

The units of each term in the previous equation are

$$\frac{C_0}{T} - \frac{DC_0}{x^2}\left(= \frac{DC_0}{2DT}\right) = -\frac{\tau^{1/2}\mathcal{D}_{\ni}\mathcal{C}_{,}}{(2DT)^{3/2}} + \frac{\tau\mathcal{D}_{\triangle}\rfloor_{,}}{(2DT)^2} - \frac{\tau C_0}{2T^2}$$

If we now rewrite this in terms on the dimensionless variables:

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} - \frac{1}{2}\frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} = -\epsilon^{1/2}\frac{\mathcal{D}_{\ni}}{(2D)^{3/2}}\frac{\partial^3 \tilde{C}}{\partial \tilde{x}^3} + \epsilon\left(\frac{\mathcal{D}_{\triangle}}{(2D)^2}\frac{\partial^4 \tilde{C}}{\partial \tilde{x}^4} - \frac{1}{2}\frac{\partial^2 \tilde{C}}{\partial \tilde{t}^2}\right) + O(\epsilon^{3/2})$$

We now want to use recursively

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} = \frac{1}{2}\frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} + O(\epsilon^{1/2})$$

to rewrite the second time derivative

$$\begin{aligned}
\frac{\partial^2 \tilde{C}}{\partial \tilde{t}^2} &= \frac{\partial}{\partial \tilde{t}}\left(\frac{1}{2}\frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2}\right) + O(\epsilon^{1/2}) \\
&= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial \tilde{C}}{\partial \tilde{t}} + O(\epsilon^{1/2})\right) \\
&= \frac{1}{4}\frac{\partial^4 \tilde{C}}{\partial \tilde{x}^4} + +O(\epsilon^{1/2})
\end{aligned}$$

It is thus possible to write every term on the right hand side as a spatial derivative plus higher order terms. The coefficients of each order become increasingly complicated, but the first few are simple enough (and for our purposes today this is all we care about). We therefore have:

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} - \frac{1}{2}\frac{\partial^2 \tilde{C}}{\partial \tilde{x}^2} = -\epsilon^{1/2}\frac{\lambda_3}{3!}\frac{\partial^3 \tilde{C}}{\partial \tilde{x}^3} + \epsilon\frac{\lambda_4}{4!}\frac{\partial^4 \tilde{C}}{\partial \tilde{x}^4} - \epsilon^{3/2}\frac{\lambda_5}{5!}\frac{\partial^5 \tilde{C}}{\partial \tilde{x}^5} + O(\epsilon^2)$$

where the first few λ_i are, in terms of the moments M_i,

$$\begin{aligned}
\lambda_1 &= \mu \\
\lambda_2 &= \sigma^2 \\
\lambda_3 &= \frac{3!\mathcal{D}_{\ni}}{(2D)^{3/2}} = \frac{M_3}{M_2^{3/2}} \\
\lambda_4 &= 4!\left(\frac{\mathcal{D}_{\triangle}}{(2D)^2} - \frac{1}{8}\right) = \frac{M_4}{M_2^2} - 3 = \kappa.
\end{aligned}$$

These λ_i are called **normalized cumulants**, and they give an indication of the "distance" between the distribution of the steps and a gaussian distribution. They are termed cumulants because if one sums a random variable with a certain distribution, the cumulants are additive and so the cumulant of the sum is the sum of the cumulants. The third cumulant measures the skewness of the distribution. The fourth one is called Kurtosis and is in general denoted by κ.

2.2 Perturbation Analysis Using Similarity Variables

There are two ways to proceed from here. The first one is to make a similarity variable assumption and a perturbation expansion. More precisely, this means changing from the variables $(\tilde{x}, \tilde{t})$ to $(\zeta, \tilde{t})$ where

$$\zeta = \frac{\tilde{x}}{\sqrt{\tilde{t}}} = \frac{x}{\sqrt{2Dt}}$$

In term of those variable, one now lets $\tilde{C}(\tilde{x}, \tilde{t}) = \phi(\zeta, \tilde{t})/\sqrt{\tilde{t}}$. to get the equation:

$$-2\tilde{t}\frac{\partial \phi}{\partial \tilde{t}} + \left(\frac{\partial^2 \phi}{\partial \zeta^2} + \zeta\frac{\partial \phi}{\partial \zeta} + \phi\right) = -\epsilon^{1/2}\frac{\lambda_3}{3!}\frac{\partial^3 \phi}{\partial \zeta^3} + \epsilon\frac{\lambda_4}{4!}\frac{\partial^4 \phi}{\partial \zeta^4} + O(\epsilon^{3/2})$$

Notice that if we divide on both sides by $\tilde{t}$, the time scale we introduced quite arbitrarly at the beginning, T, drops out of the expression, The pertubation expansion needed here is of the form:

$$\phi(\zeta, \tilde{t}) = \phi_0(\zeta) + \frac{\epsilon^{1/2}}{\tilde{t}^{1/2}}\phi(\zeta) + \frac{\epsilon}{\tilde{t}}\phi_1(\zeta)$$

This is then equivalent to what was done previously, with $\xi = \tau/t$. One now takes the limit $\xi \to 0$ and ζ fixed. However, using Fourier transforms should save us some work.

2.3 Perturbation Analysis Using Fourier Transforms

Consider the Fourier transform in $\tilde{x}$

$$\hat{C}(\tilde{k}, \tilde{t}) = \int_{-\infty}^{\infty} e^{-i\tilde{k}\tilde{x}}\tilde{C}(\tilde{x}, \tilde{t})d\tilde{x}$$

and let us use the relation

$$\widehat{\frac{\partial^n \tilde{C}}{\partial \tilde{x}^n}} = (i\tilde{k})^n \tilde{C}$$

to rewrite the equation in terms of the Fourier variables

$$\frac{\partial \hat{C}}{\partial \tilde{t}} = \left(\frac{(i\tilde{k})^2}{2!} - \epsilon^{1/2}\frac{\lambda_3(i\tilde{k})^3}{3!} + \epsilon\frac{\lambda_4(i\tilde{k})^4}{4!} + O(\epsilon^{3/2})\right)\hat{C}(\tilde{k}, \tilde{t})$$

The coefficient on the right hand side can be seen as a Taylor series of a function $-\psi(\tilde{k}, \epsilon)$ such that we have:

$$\frac{\partial \hat{C}}{\partial \tilde{t}} = -\psi(\tilde{k}, \epsilon)\hat{C}(\tilde{k}, \tilde{t})$$

The solution to this equation is very simple and is $\hat{C}(\tilde{k},\tilde{t}) = \hat{C}(\tilde{k},0)e^{-\psi(\tilde{k},\epsilon)\tilde{t}}$. We now simply have to take the inverse transform to find the desired function. Notice that up to this point, this solution is still exact.

$$\tilde{C}(\tilde{x},\tilde{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{C}(\tilde{k},0)e^{i\tilde{k}\tilde{x}-\psi(\tilde{k},\epsilon)\tilde{t}}d\tilde{k}$$

However, unless we could find explicitly what ψ is, we have to use its expansion in $\tilde{k}$ which constrains us to values of $\tilde{k}$ close to 0. Let us assume that the initial condition is a delta function at $\tilde{x} = 0$ such that $\tilde{C}(\tilde{k},0) = 1$. The Fourier transform is then

$$\hat{C}(\tilde{k},\tilde{t}) = e^{\tilde{t}\left(\frac{(i\tilde{k})^2}{2!} - \epsilon^{1/2}\frac{\lambda_3(i\tilde{k})^3}{3!} + \epsilon\frac{\lambda_4(i\tilde{k})^4}{4!} + O(\epsilon^{3/2})\right)}$$

If we now change variables and let $\omega = \tilde{k}\sqrt{\tilde{t}}$ and $\zeta = \frac{\tilde{x}}{\sqrt{\tilde{t}}} = \frac{x}{\sqrt{2Dt}}$, we note that the time scale that was artificially introduced, T, does not appear in ζ. We are left with

$$\tilde{C}(\tilde{x},\tilde{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\zeta}e^{-\omega^2/2}e^{-(i\omega)^3\frac{\lambda_3\epsilon^{1/2}}{3!\tilde{t}^{1/2}} + (i\omega)^4\frac{\lambda_4\epsilon}{4!\tilde{t}} + O((\epsilon/\tilde{t})^{3/2})}$$

We now define the small parameter $\xi = \epsilon/\tilde{t} = \tau/t$ and use a Taylor expansion of the third exponential in the previous expression

$$e^{-(i\omega)^3\frac{\lambda_3}{3!}(\frac{\epsilon}{\tilde{t}})^{1/2} + (i\omega)^4\frac{\lambda_4\epsilon}{4!\tilde{t}} + O((\epsilon/\tilde{t})^{3/2})} = 1 - \frac{(i\omega)^3}{3!}\lambda_3\xi^{1/2} + \left(\frac{(i\omega)^4}{4!}\lambda_4 + \frac{(i\omega)^6}{2(3!)^2}\lambda_3^2\right) + O(\xi^{3/2})$$

We also need to use the Gaussian integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\zeta}e^{-\omega^2/2}d\omega = \frac{e^{-\zeta^2/2}}{\sqrt{2\pi}}$$

and use again the fact that multiplying by $(i\omega)$ is equivalent to taking derivatives with respect to ζ. Combining all those, we get

$$\tilde{C}(\tilde{x},\tilde{t}) = \frac{1}{\sqrt{\tilde{t}}}\left(1 - \xi^{1/2}\frac{\lambda_3}{3!}\frac{\partial^3}{\partial\zeta^3} + \xi\left(\frac{\lambda_4}{4!}\frac{\partial^4}{\partial\zeta^4} + \frac{\lambda_3^2}{2(3!)^2}\frac{\partial^6}{\partial\zeta^6}\right) + O(\xi^{3/2})\right)\frac{e^{-\zeta^2/2}}{\sqrt{2\pi}}$$

If we write explicitly the derivatives of the Gaussian:

$$\begin{aligned}
\phi_0 &= \frac{e^{-\zeta^2/2}}{\sqrt{2\pi}} \\
\phi_0' &= -\zeta\phi_0 \\
\phi_0'' &= (\zeta^2 - 1)\phi_0 \\
\phi_0^{(3)} &= (3\zeta - \zeta^3)\phi_0 \\
\phi_0^{(4)} &= (3 - 6\zeta^2 + \zeta^4)\phi_0 \\
\phi_0^{(5)} &= (-15\zeta + 10\zeta^3 - \zeta^5)\phi_0 \\
\phi_0^{(6)} &= (-15 + 45\zeta^2 - 15\zeta^4 + \zeta^6)\phi_0
\end{aligned}$$

we can have explicitly the first corrections

$$\tilde{C}(\tilde{x},\tilde{t}) = \frac{e^{-\zeta^2/2}}{\sqrt{2\pi\tilde{t}}}\left(1+\xi^{1/2}f_{1/2}(\zeta)+\xi f_1(\zeta)+O(\xi^{3/2})\right)$$

where

$$\begin{aligned} f_{1/2}(\zeta) &= -\frac{\lambda_3}{3!}(3\zeta-\zeta^3) \\ f_1(\zeta) &= \left(\frac{\lambda_4}{8}-\frac{5\lambda_3^2}{24}\right)-\left(\frac{\lambda_4}{4}-\frac{5\lambda_3^2}{8}\right)\zeta^2+\left(\frac{\lambda_4}{24}-\frac{5\lambda_3^2}{24}\right)\zeta^4+\frac{\lambda_3^2}{72}\zeta^6. \end{aligned}$$

This kind of expansion about the leading-order Gaussian dependence of the PDF for a sum of a large nubmer of iid random variables is sometimes called a **Gram-Charlier expansion**.

2.4 Edge of the Central Region

From this expression, we can deduce the width of the "central region" over which the leading-order Gaussian approximation is valid. When the highest in power of ζ becomes of order one, the correction term is of the same order as the gaussian so the zeroth order solution is not valid anymore. This happens when $\lambda_3\zeta^3\xi^{1/2}\approx 3 \Rightarrow \zeta_{edge}\approx(3!/\lambda_3)^{1/3}\xi^{-1/2}$, or in terms of the dimensional variables,

$$\begin{aligned} \frac{x_{edge}}{\sqrt{2Dt}} &\sim \left(\frac{t}{\tau}\right)^{1/6}\left(\frac{3!}{\lambda_3}\right)^{1/3} \\ x_{edge} &\sim \sqrt{2D}\frac{t^{2/3}}{\tau^{1/6}}\left(\frac{3!}{\lambda_3}\right)^{1/3} \\ x_{edge} &\sim \sigma\left(\frac{3!}{\lambda_3}\right)^{1/3} n^{2/3} \end{aligned}$$

where x_{edge} is the position at which the gaussian is still valid, n is the number of steps, and $\sigma^2 = 2D\tau$ is the variance of each step. This results is obtained assuming that λ_3 is non-zero. It is worth noting that all the corrections become of order one at the same point since the highest power of ζ always yields a term of the form $\zeta^{3i}\xi^{i/2}$. In the case $\lambda_3 = 0$ and $\lambda_4 = \kappa \neq 0$, the central region becomes wider ($x_{edge}\sim n^{3/4}$) and in general as more $\lambda_n = 0$, the central region becomes wider. This simply expresses the fact that the distribution of the steps is close to a gaussian, for which there is no correction.

2.5 Additivity of Cumulants

Finally, from the form of the Fourier transform

$$\hat{C}(\tilde{k},\tilde{t}) = e^{\tilde{t}\left(\frac{(i\tilde{k})^2}{2!}-\epsilon^{1/2}\frac{\lambda_3(i\tilde{k})^3}{3!}+\epsilon\frac{\lambda_4(i\tilde{k})^4}{4!}+O(\epsilon^{3/2})\right)}$$

one can get some more understanding about the cumulants. Since the time variable can be written as $\tilde{t} = n\tau/T$, the above expression shows that summing n times the same random variables simply multiplies by n the cumulants. The linearity in $\tilde{t}$ is basically equivalent to the fact that the cumulants are additive.