Solutions to Problem Set 2a

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1 Inelastic Diffusion

Given a distribution of iid random variables, ΔX_n , with cumulants, $\tilde{c}_{n,l}$, which describe the pdf for displacement in a single step, we seek to describe the result of rescaling ΔX_n by an exponentially decreasing factor with each step, such that the pdf for position after N steps is,

$$X_N = \sum_{n=1}^{N} a^n \Delta X_n \qquad 0 < a < 1 \tag{1}$$

The probability distribution of a sum of iid random variables is given by a convolution of the distributions of each random variable. In our case,

$$P_N(X) = p_1(x) * p_2(x) * \cdots * p_n(x), \tag{2}$$

where $p_n(x)$ are the rescaled distributions for each step. Because of the convolution theorem, the Fourier transform of the distribution after N steps, $\hat{P}_N(k)$, can be expressed as a product of the Fourier transforms of the individual step distributions:

$$\hat{P}_N(k) = \hat{p}_1(k)\hat{p}_2(k)\ldots\hat{p}_1(k) \tag{3}$$

By inverse Fourier transform, the probability distribution is obtained:

$$P(X) = \int_{-\infty}^{\infty} e^{ikx}\hat{p}_1(k)\hat{p}_2(k)\ldots\hat{p}_1(k)dx \tag{4}$$

Going further, we can express $\hat{P}_N(k)$ in terms of the cumulants of the individual steps, which are given by $c_{n,l} = (a^n)^l \tilde{c}_{n,l}$. Because cumulants of $p_n(x)$ are additive, the cumulants of $P_N(x)$ are given by,

$$C_{N,l} = i^l \frac{d^l}{dk^l} \log \hat{P}_N(k\ )|_{k=0} = \sum_{n=1}^{N} (a^n)^l \tilde{c}_{n,l} \tag{5}$$

It turns out that the latter geometric series is convergent, since, generally speaking, for any such series,

$$\sum_{n=1}^{N} a^n = \frac{1-a^{N+1}}{1-a} - 1 = \frac{a-a^{N+1}}{1-a} \qquad |a| < 1 \tag{6}$$

and thus,

$$C_{N,l} = \tilde{c}_{n,l}\left(\frac{a^l - a^{l(N+1)}}{1-a^l}\right) \tag{7}$$

Since by definition,

$$\log \hat{P}_N(k) = \sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} C_{N,l} = \sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} \tilde{c}_{n,l}\left(\frac{a^l - a^{l(N+1)}}{1-a^l}\right), \tag{8}$$

An expression for $\hat{P}_N(k)$ is obtained in terms of the original cumulants:

$$\hat{P}_N(k) = e^{\sum\limits_{l=0}^{\infty} \frac{(-ik)^l}{l!} \tilde{c}_{n,l}\left(\frac{a^l - a^{l(N+1)}}{1-a^l}\right)} \tag{9}$$

Finally, taking the inverse transform yields the probability distribution,

$$P(X) = \int_{-\infty}^{\infty} e^{ikx} e^{\sum\limits_{l=0}^{\infty} \frac{(-ik)^l}{l!} \tilde{c}_{n,l}\left(\frac{a^l - a^{l(N+1)}}{1-a^l}\right)} dx \tag{10}$$

(C) Let $N \to \infty$, and define $C_l = \lim_{N\to\infty} C_{N,l}$. Utilizing (5), $C_l = \sum\limits_{n=1}^{\infty} a^{nl}\tilde{c}_{n,l}$, and taking advantage of the convergence of the geometric series, C_l can be written as,

$$C_l = \tilde{c}_{n,l}\left(\frac{a^l}{1-a^l}\right) \tag{11}$$

If ϵ is then defined such that $a = 1-\epsilon \quad 0 < \epsilon < 1$, then C_l becomes

$$C_l = \tilde{c}_{n,l}\left(\frac{(1-\epsilon)^l}{1-(1-\epsilon)^l}\right) \tag{12}$$

When the above expression is expanded for any l, the result is that $C_l = O\left(\epsilon^{-1}\right)$ for any l. For example,

$$C_2 = \tilde{c}_{n,2}\left(\frac{(1-\epsilon)^2}{1-(1-\epsilon)^2}\right) = \tilde{c}_{n,2}\left(\frac{1-2\epsilon+\epsilon^2}{2\epsilon-\epsilon^2}\right) \cong \tilde{c}_{n,2}\left(\frac{1}{2\epsilon}\right) = O\left(\epsilon^{-1}\right) \tag{13}$$

It then follows that

$$\frac{C_{2m}}{C_2^m} \cong \left(\frac{\tilde{c}_{2m}\epsilon^{-1}}{\tilde{c}_2^m \epsilon^{-m}}\right) = O\left(\epsilon^{m-1}\right). \tag{14}$$

(D) From the definition of cumulants (5), we can explicitly write the first few terms in the Taylor expansion of $\log \hat{P}_{N\to\infty}(k)$ around $k=0$:

$$\log \hat{P}_{N\to\infty}(k) = \sum_{l=0}^{\infty} \frac{(-ik)^l}{l!} C_l = \frac{-k^2}{2} C_2 + \frac{ik^3}{3!} C_3 - \frac{k^4}{4!} C_4 + ... \tag{15}$$

Thus,

$$\hat{P}_{N\to\infty}(k) = e^{-\frac{k^2}{2}C_2+\frac{ik^3}{3!}C_3-\frac{k^4}{4!}C_4+\dots} \tag{16}$$

and taking the inverse Fourier transform gives,

$$P_{N\to\infty}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-\frac{k^2}{2}C_2}e^{\frac{ik^3}{3!}C_3-\frac{k^4}{4!}C_4+\dots}dk. \tag{17}$$

The following changes of variable are now made:

$$\omega = k\sqrt{C_2} \qquad \zeta = \frac{X}{\sqrt{C_2}} \tag{18}$$

and the third exponential above is expanded in a Taylor series by utilizing

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{19}$$

to obtain,

$$P_{N\to\infty}(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\zeta}e^{-\frac{\omega^2}{2}}\left(1+\frac{ik^3}{3!}C_3 - \frac{k^4}{4!}C_4 + \dots\right)dk \tag{20}$$

Since, from (14), $\frac{C_l}{C_2^{l/2}} = O(\epsilon^{l/2-1})$,

$$P_{N\to\infty}(X) = \frac{1}{2\pi\sqrt{C_2}}\int_{-\infty}^{\infty} e^{i\omega\zeta}e^{-\frac{\omega^2}{2}}\left(1+O(\epsilon^{1/2})\right)dk \underset{\epsilon\to 0}{=} \frac{1}{2\pi\sqrt{C_2}}\int_{-\infty}^{\infty} e^{i\omega\zeta}e^{-\frac{\omega^2}{2}}dk \tag{21}$$

which is simply the Gaussian integral with the addition of $\sqrt{C_2}$ in the denominator. Upon integration, an expression for P_∞ is obtained:

$$P_{N\to\infty,\epsilon\to 0}(X) = \frac{e^{-X^2/2}}{\sqrt{2\pi\sigma^2}} \qquad \sigma^2 = C_2 = \tilde{c}_{n,2}\left(\frac{(1-\epsilon)^2}{1-(1-\epsilon)^2}\right) \underset{\epsilon\to 0}{=} \tilde{c}_{n,2}\left(\frac{1}{2\epsilon}\right) \tag{22}$$

If we then define

$$\phi(\zeta) = C_2^{1/2}P_{N\to\infty,\varepsilon\to 0}(\zeta C_2^{1/2}), \tag{23}$$

and substitute into (22), we find that

$$\phi(\zeta) = C_2^{1/2}P_{N\to\infty,\varepsilon\to 0}(\zeta C_2^{1/2}) = \frac{e^{-\zeta^2/2}}{\sqrt{2\pi}}, \tag{24}$$

which shows that the Central Limit Theorem holds.

(E) Gram-Charlier Expansion.

Return to the expansion of $\log \hat{P}_{N\to\infty}(k)$ in terms of the cumulants (15), and the resulting inverse Fourier transform expression for $P_{N\to\infty}(X)$ (17). Again utilizing the exponential expansion (19), we carry out the expansion of the third exponential in (17) to higher order:

$$P_{N\to\infty}(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-\frac{k^2}{2}C_2}\left[1-\frac{(ik)^3}{3!}C_3+\frac{(ik)^4}{4!}C_4-\frac{(ik)^5}{5!}C_5+\left(\frac{C_6}{6!}+\frac{C_3^3}{(2)(3!)}\right)(ik)^6\right.$$

$$-\left(\frac{C_7}{7!}+\frac{C_3C_4}{3!4!}\right)(ik)^7+\left(\frac{C_8}{8!}+\frac{C_4^2}{(4!)^2}+\frac{C_3C_5}{3!5!}\right)(ik)^8+...]dk \tag{25}$$

Note that the expansion of (15) has been altered from orders of $(-i\omega)^l$ to orders of $(i\omega)$. Now, making the usual change of variable (18), the inverse Fourier transform expression for $P_{N\to\infty}(X)$ is obtained:

$$P_{N\to\infty}(X)=\frac{1}{2\pi C_2}\int_{-\infty}^{\infty}e^{i\omega\zeta}e^{-\frac{\omega^2}{2}}\left[1-\frac{(i\omega)^3}{3!}\frac{C_3}{C_2^{3/2}}+\frac{(i\omega)^4}{4!}\frac{C_4}{C_2^2}-\frac{(i\omega)^5}{5!}\frac{C_5}{C_2^{5/2}}+\left(\frac{C_6}{6!}+\frac{C_3^2}{(2)(3!)}\right)\frac{(i\omega)^6}{C_2^3}\right.$$

$$-\left(\frac{C_7}{7!}+\frac{C_3C_4}{3!4!}\right)\frac{(i\omega)^7}{C_2^{7/2}}+\left(\frac{C_8}{8!}+\frac{C_4^2}{(4!)^2}+\frac{C_3C_5}{3!5!}\right)\frac{(i\omega)^8}{C_2^4}+...]d\omega \tag{26}$$

As $\epsilon\to 0$, it is easy to see that successive terms in the series of correction terms above are increasing in powers of ϵ, since $\frac{C_l}{C_2^{l/2}}=O(\epsilon^{l/2-1})$. Thus,

$$P_{N\to\infty}(X)\cong\frac{1}{2\pi C_2}\int_{-\infty}^{\infty}e^{i\omega\zeta}e^{-\frac{\omega^2}{2}}\left[1-\frac{(i\omega)^3}{3!}\frac{\tilde{c}_3}{\tilde{c}_2^{3/2}}\epsilon^{1/2}+\left(\frac{(i\omega)^4}{4!}\frac{\tilde{c}_4}{\tilde{c}_2^2}+\frac{(i\omega)^6}{(2)(3!)}\frac{\tilde{c}_3^2}{\tilde{c}_2^3}\right)\epsilon\right.$$

$$-\left(\frac{(i\omega)^5}{5!}\frac{\tilde{c}_5}{\tilde{c}_2^{5/2}}+\frac{(i\omega)^7}{3!4!}\frac{\tilde{c}_3\tilde{c}_4}{c_2^{7/2}}\right)\epsilon^{3/2}+\left(\frac{(i\omega)^6}{6!}\frac{\tilde{c}_6}{\tilde{c}_2^3}+\frac{(i\omega)^8}{(4!)^2}\frac{\tilde{c}_4^2}{\tilde{c}_2^4}+\frac{(i\omega)^8}{3!5!}\frac{\tilde{c}_3\tilde{c}_5}{c_2^4}\right)\epsilon^2+O(\epsilon^{5/2})]d\omega \tag{27}$$

However, this holds only in $\lim_{\epsilon\to 0}$. Recalling (13) it can be seen that higher orders of ϵ enter into the picture when calculating the $\epsilon-$dependence of C_l for larger values of ϵ. Thus terms containing lower leading order $\tilde{c}_l$ in (27) are carried over to higher order. The details get rather complicated, especially since higher order terms in the denominator of (13) may also matter. Considering that the error in neglecting ϵ^2 terms is $O(\epsilon)$, the second order terms can be significant. The best thing, it seems, is to recognize by (27) that the correction terms are decreasing in powers of ϵ at leading order and be assured of the convergence of the series to a reliable solution, but to utilize (26) to calculate the correction terms.

Carrying out the integration in (26), keeping only the terms to $O(\epsilon^2)$ in (27), and utilizing the fact that multiplying by $(i\omega)^n$ is equivalent to taking the n^{th} derivative of the Gaussian, $(i\omega)^n=\frac{\partial^n}{\partial\zeta^n}$ we arrive at the Gram-Charlier expansion for $P_{N\to\infty}(X)$:

$$P_{N\to\infty}(X)=\frac{1}{\sqrt{2\pi C_2}}\left[1-\frac{\lambda_3}{3!}\frac{\partial^3}{\partial\zeta^3}+\frac{\lambda_4}{4!}\frac{\partial^4}{\partial\zeta^4}-\frac{\lambda_5}{5!}\frac{\partial^5}{\partial\zeta^5}+\left(\frac{\lambda_6}{6!}+\frac{1}{2(3!)^2}\frac{C_3^2}{C_2^3}\right)\frac{\partial^6}{\partial\zeta^6}\right.$$

$$-\frac{1}{3!4!}\frac{C_3C_4}{C_2^{7/2}}\frac{\partial^7}{\partial\zeta^7}+\left(\frac{1}{2(4!)}\frac{C_4^2}{C_2^4}+\frac{1}{2(4!)}\frac{C_3C_5}{C_2^4}\right)\frac{\partial^8}{\partial\zeta^8}]e^{-\zeta^2/2} \tag{28}$$

where $\lambda_l=\frac{C_l}{C_2^{l/2}}$.

The first eight derivatives of the Gaussian are:

$$\begin{aligned}
\phi_o &= e^{-\zeta^2/2} \equiv P_o \\
\phi_o^{'} &= -\zeta\phi_o \equiv P_1 \\
\phi_o^{''} &= (\zeta^2-1)\phi_o \equiv P_2 \\
\phi_o^{(3)} &= (3\zeta-\zeta^3)\phi_o \equiv P_3 \\
\phi_o^{(4)} &= (3-6\zeta^2+\zeta^4)\phi_o \equiv P_4 \\
\phi_o^{(5)} &= (15\zeta+10\zeta^3-\zeta^5)\phi_o \equiv P_5 \\
\phi_o^{(6)} &= (-15+45\zeta^2-10\zeta^4+\zeta^6)\phi_o \equiv P_6 \\
\phi_o^{(7)} &= (-105\zeta-105\zeta^3+21\zeta^5-\zeta^7)\phi_o \equiv P_7 \\
\phi_o^{(8)} &= (105-420\zeta^2+210\zeta^4-28\zeta^6-\zeta^8)\phi_o \equiv P_8
\end{aligned}$$

And so (28) becomes,

$$P_{N\to\infty}(X) = \frac{1}{\sqrt{2\pi C_2}}\left[P_o - \frac{\lambda_3}{3!}P_3 + \frac{\lambda_4}{4!}P_4 - \frac{\lambda_5}{5!}P_5 + \left(\frac{\lambda_6}{6!} + \frac{1}{2(3!)^2}\frac{C_3^2}{C_2^3}\right)P_6 \right.$$
$$\left. -\frac{1}{3!4!}\frac{C_3C_4}{C_2^{7/2}}P_7 + \left(\frac{1}{2(4!)}\frac{C_4^2}{C_2^4} + \frac{1}{2(4!)}\frac{C_3C_5}{C_2^4}\right)P_8\right] \tag{29}$$

(F) Simulation of exponentially decaying simple walk.

We will now compare simulations of an exponentially decaying walk to the Gram-Charlier expansion for different values of ϵ. But first, the cumulants of the simple random walk, $\tilde{c}_l$ must be determined.

For a simple random walk, the original probability distribution is given by,

$$\tilde{p}_n(x) = \frac{1}{2}\left[\delta(x-1)+\delta(x+1)\right]$$

and the Fourier transform is $\hat{p}_n(k) = \cos(k)$. Taylor expanding $\hat{p}(k)$ utilizing the cosine expansion gives

$$\hat{p}_n(k) = 1 - \frac{k^2}{2!} + \frac{k^4}{4!} - \frac{k^6}{6!} + ... \tag{30}$$

Then, utilizing the logarithmic Taylor expansion,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

and collecting terms, we can write,

$$\log\hat{p}_n(k) = -\frac{k^2}{2} - \frac{2k^4}{4!} - \frac{16k^6}{6!} + O(k^8) \tag{31}$$

By virtue of the definition of C_l, (15), the first three cumulants of both the original and the rescaled distributions are now known:

$$\tilde{c}_2 = 1 \qquad \tilde{c}_4 = 2 \qquad \tilde{c}_6 = 16$$

$$C_2 = \tfrac{(1-\epsilon)^2}{1-(1-\epsilon)^2} \qquad C_4 = \tfrac{2(1-\epsilon)^4}{1-(1-\epsilon)^4} \qquad C_6 = \tfrac{16(1-\epsilon)^6}{1-(1-\epsilon)^6}$$

and can be substituted into (29) to obtain the corrected Gaussian distribution (to $O(\epsilon^2)$ for this analysis).

Figures $1-10$ show the results of simulations compared with the corresponding Gaussian and corrected-Gaussian (to $O(\epsilon^2)$) predictions for $\epsilon = 0.5$, $1-\frac{1}{\sqrt{2}}$, 0.25, 0.1, and 0.01. The exact solutions are known for the first two cases and are also included in the plots. For $\epsilon = 0.5$ the probability distribution is the uniformly distributed function,

$$P_{N\to\infty}(X) = \begin{cases} 1/2 & -1 < X < 1 \\ 0 & otherwise \end{cases}$$

In the case of $\epsilon = 1-\frac{1}{\sqrt{2}}$, the following convolution gives the exact result:

$$P(X, a = \frac{1}{\sqrt{2}}) = P(\sqrt{2}X, 1/2) * P(X, 1/2).$$

2 Normal Diffusion with Fat-Tailed Transitions

(A) Given the transition probability for a random flight,

$$p\,(x) = \frac{A}{(1+x^2)^2}, \tag{32}$$

the "student t" distribution with a power law tail, we seek an expression for the structure function, $\hat{p}(k)$ and the Fourier transform of the position after n steps, $\hat{P}_n(k)$.

The Fourier transform is given by,

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{-ikx} p(x) = \int_{-\infty}^{\infty} e^{-ikx} \frac{A}{(1+x^2)^2} dx = \int_{-\infty}^{\infty} e^{-ikx} \frac{A}{(x+i)^2 (x-i)^2} dx \tag{33}$$

The integrand is now defined in terms of the complex variable, z, as

$$f(z) \equiv e^{-ikz} \frac{A}{(z+i)^2 (z-i)^2}. \tag{34}$$

We notice that $f(z)$ has double poles of order two at $\pm i$. We also notice that $\hat{p}(k)$ is an even function since

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{-ikx} \frac{A}{(1+x^2)^2} = \int_{-\infty}^{\infty} \frac{\cos(kx)}{(1+x^2)^2} dx - i \int_{-\infty}^{\infty} \frac{\sin(kx)}{(1+x^2)^2} dx,$$

and the second integral is an odd function and therefore zero, leaving just the first, even, integral. Because $\hat{p}(k)$ is even, $\hat{p}(k) = \hat{p}(-k)$, and thus we have the freedom to choose the contour of integration to be either the upper or lower half plane. Choosing a contour in the lower half plane from $-R$ to R on the real axis and considering only $k > 0$, we will use the

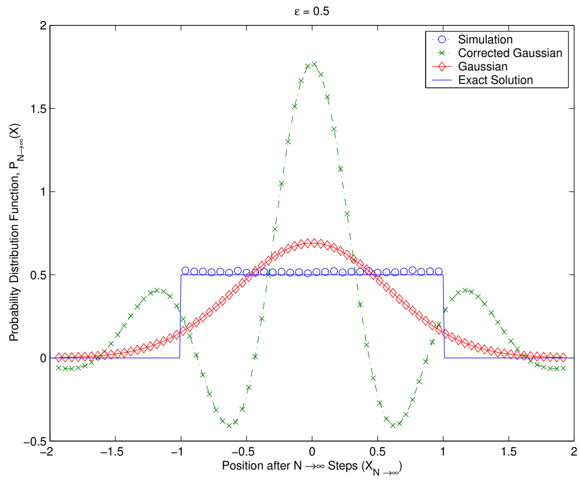

Figure 1: Simulation, $\epsilon = 0.5$

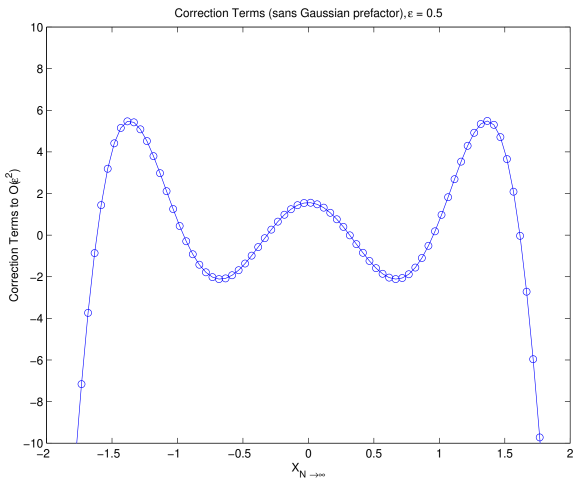

Figure 2: Correction Terms,$\epsilon = 0.5$

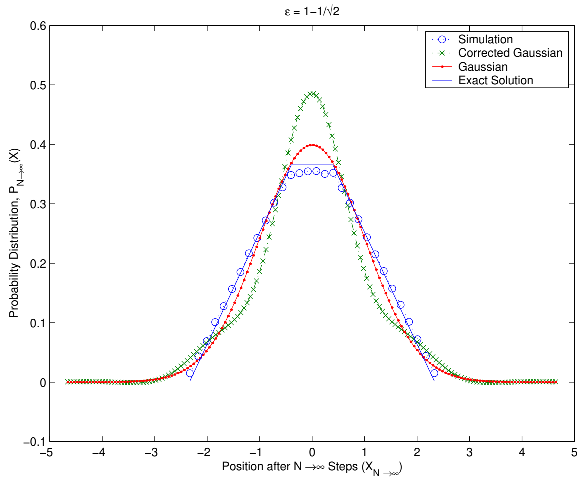

Figure 3: Simulation, $\epsilon = 1 - \frac{1}{\sqrt{2}}$

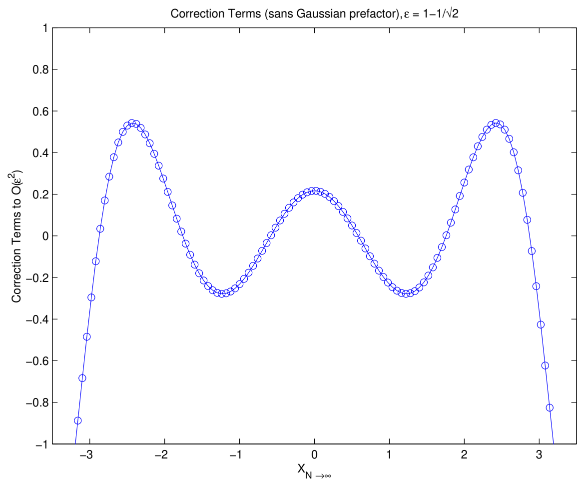

Figure 4: Correction Terms, $\epsilon = 1 - \frac{1}{\sqrt{2}}$

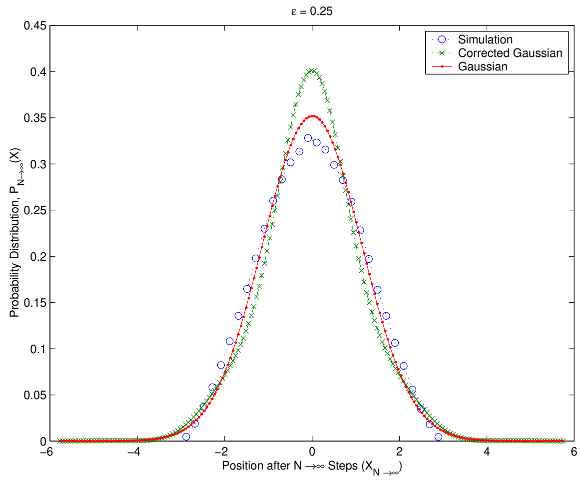

Figure 5: Simulation, $\epsilon = 0.25$

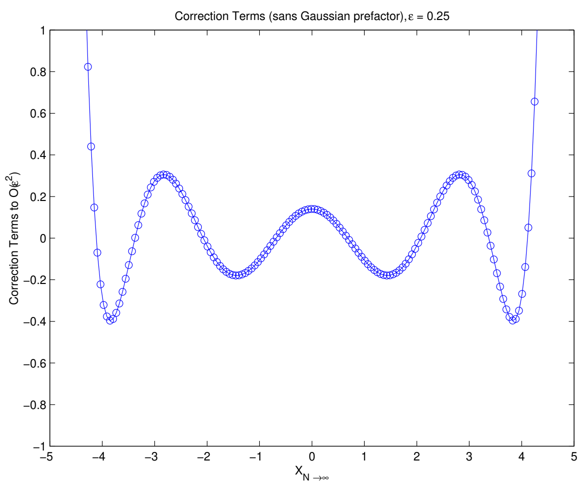

Figure 6: Correction Terms, $\epsilon = 0.25$

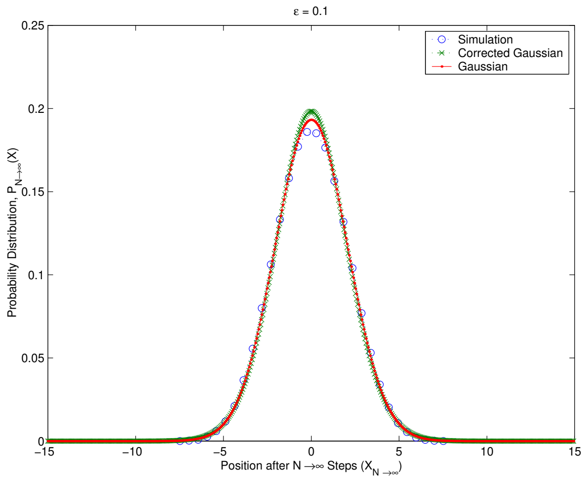

Figure 7: Simulation, $\epsilon = 0.1$

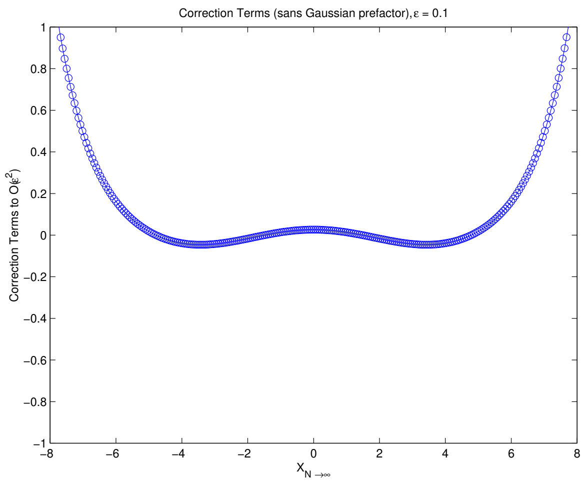

Figure 8: Correction Terms, $\epsilon = 0.1$

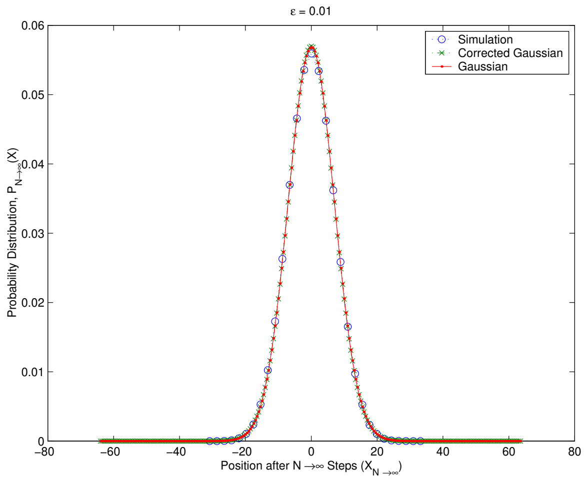

Figure 9: Simulation, $\epsilon = 0.01$

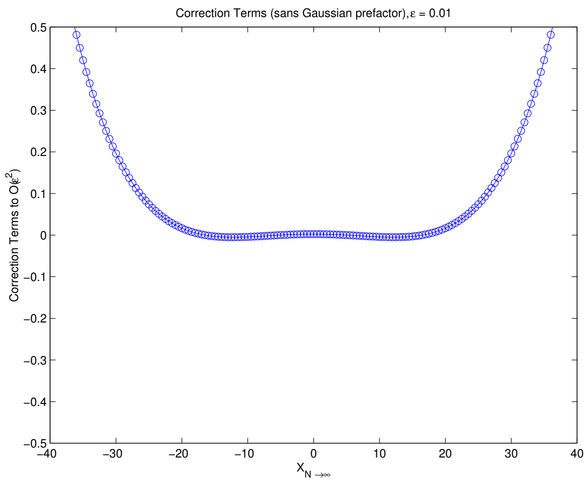

Figure 10: Correction Terms, $\epsilon = 0.01$

residue of $f(z)$ at $-i$ to compute the integral. The integral over the closed contour, C is written

$$\oint_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = -2\pi i Res(-i), \tag{35}$$

the negative sign appearing in front of the residue because by integrating from $-R$ to R, the contour is traversed in the clockwise (negative) direction. By (35) it is seen that to compute the integral we desire, the integral over C_R must vanish as $R \to \infty$:

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz \underset{R\to\infty}{=} \int_{-R}^{R} f(z)dz = -2\pi i \mathrm{Res}(-i). \tag{36}$$

To show this, we should find a maximum bound for $\int_{C_R} f(z)dz$ and show that this bound tends to zero as $R \to \infty$. The upper bound of $|f(z)|$ on C_R, noticing that on C_R, $z = Re^{i\theta}$, is

$$|f(z)| = \left|e^{-ikz}\right| \left|\frac{A}{(1+x^2)^2}\right| = \left|e^{-ikR(\cos\theta + i\sin\theta)}\right| \left|\frac{A}{(1+R^2)^2}\right| = \left|e^{kR\sin\theta}\right| \left|\frac{A}{(1+R^2)^2}\right|.$$

Since in the lower half plane, $\sin\theta$ is negative, and since we are considering $k > 0$,

$$\left|e^{kR\sin\theta}\right| \le 1,$$

and the upper bound of $f(z)$ is $\left|\frac{A}{(1+R^2)^2}\right|$. It then follows that, since the length of the path of integration over the lower half plane is $L(C_R) = \pi R$, the upper bound of the integral over C_R is given by

$$\left|\int_{C_R} f(z)dz\right| = \max|f(z)|\, L(C_R) = \frac{\pi R A}{(1+R^2)^2},$$

which tends to zero as $R \to \infty$.

We can now compute $Res(-i)$ to obtain our desired integral. For a pole of order m at a,

$$Res(a) = \frac{1}{(m-1)!}\left[\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}\right]_{z=a},$$

so $Res(-i)$ is given by,

$$\mathrm{Res}(-i) = \frac{d}{dz}\left[(z+i)^2 e^{-ikz}\frac{A}{(z+i)^2(z-i)^2}\right]_{z=a} = \left[-\frac{Aike^{-ikz}}{(z-i)^2} - \frac{2Ae^{-ikz}}{(z-i)^3}\right]_{z=-i} = -\frac{Ae^{-k}(k+1)}{4i}.$$

Now returning to (35) we find that

$$\hat{p}(k) = \int_{-R}^{R} f(z)dz = -2\pi i Res(-i) = \frac{\pi}{2}Ae^{-k}(k+1) \qquad k > 0\ . \tag{37}$$

It can be seen that to achieve the proper normalization of the pdf it is necessary that

$$A = \frac{2}{\pi}. \tag{38}$$

Because $\hat{p}(k)$ is even, and $\hat{p}(k) = \hat{p}(-k)$, we obtain the structure function,

$$\hat{p}(k) = (|k| + 1)\, e^{-|k|}. \tag{39}$$

Finally, because of the convolution property of Fourier transforms of iid random variables shown in (2) and (3), the Fourier transform of the position after n steps is given by,

$$\hat{P}_n(k) = \hat{p}^n(k) = (k+1)^n\, e^{-n|k|}. \tag{40}$$

(B) To show that the Central Limit Theorem holds, first Taylor-expand $\hat{p}(k)$ around $k = 0$:

$$\hat{p}(k) = (|k| + 1)\, e^{-|k|} = (|k| + 1)\left(1 - |k| + \frac{k^2}{2!} - \frac{|k|^3}{3!} + \frac{k^4}{4!}\right) = 1 - \frac{k^2}{2} + \frac{|k|^3}{3} - \frac{k^4}{8} + O\left(|k|^5\right) \tag{41}$$

Then, utilizing the logarithmic Taylor expansion we can expand $\log \hat{p}(k)$ as,

$$\log \hat{p}(k) = -\frac{k^2}{2} + \frac{|k|^3}{3} - \frac{k^4}{4} + O\left(|k|^5\right) \tag{42}$$

From (40), we can express $\log \hat{P}_n(k)$ by

$$\log \hat{P}_n(k) = n \log \hat{p}(k) = -\frac{nk^2}{2} + \frac{n\,|k|^3}{3} - \frac{nk^4}{4} + O\left(|k|^5\right) \tag{43}$$

and as a result,

$$\hat{P}_n(k) = e^{n \log \hat{p}(k)} = e^{-\frac{nk^2}{2} + \frac{n|k|^3}{3} - \frac{nk^4}{4} + O\left(|k|^5\right)}. \tag{44}$$

Notice that by virtue of the $|k|^3$ term, the third moment diverges and thus higher moments are infinite. But that does not prevent the Central Limit Theorem from applying, so we proceed by taking the inverse Fourier transform of $P_n(k)$, separating the "Gaussian" and "non-Gaussian" parts, which gives the familiar expression for $P(X)$,

$$P(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-\frac{nk^2}{2}} e^{\frac{n|k|^3}{3} - \frac{nk^4}{4} + O\left(|k|^5\right)} dk.$$

Next we make the usual change of variable,

$$\omega = k\sqrt{n} \qquad \zeta = \tfrac{X}{\sqrt{n}}\ ,$$

and expand the third exponential inside the integral to obtain

$$P(X) = \frac{1}{2\pi\sqrt{n}} \int_{-\infty}^{\infty} e^{ik\zeta} e^{-\frac{\omega^2}{2}} e^{\frac{|\omega|^3}{3\sqrt{n}} - \frac{\omega^4}{4n} + O\left(\frac{|\omega|^5}{n^{3/2}}\right)} d\omega = \frac{1}{2\pi\sqrt{n}} \int_{-\infty}^{\infty} e^{ik\zeta} e^{-\frac{\omega^2}{2}} \left(1 + \frac{|\omega|^3}{3\sqrt{n}} - \frac{\omega^4}{4n} + O\left(\frac{|\omega|^5}{n^{3/2}}\right)\right) d\omega$$

Since higher order terms in the expansion are decreasing in negative powers of n, it is seen that

$$\lim_{n\to\infty} P_n(X) = \frac{1}{2\pi\sqrt{n}} \int_{-\infty}^{\infty} e^{ik\zeta} e^{-\frac{\omega^2}{2}} d\omega = \frac{e^{-\zeta^2}}{\sqrt{2\pi n}} \tag{45}$$

and thus the Central Limit Theorem holds. Further, the usual diffusive scaling holds, since, by substitution into (45),

$$\sigma\sqrt{n}P_n\left(\zeta\sigma\sqrt{n}\right) = \frac{e^{-\zeta^2}}{\sqrt{2\pi}},$$

with $\sigma = 1$.

3 Anomalous Diffusion with Very Fat-Tailed Transitions.

A random flight of *iid* steps with transition probability distribution

$$p(x) = \frac{A}{1+x^2},$$

the "Cauchy Distribution," has infinite variance and therefore Central Limit Theorem does not hold. We wish to find an expression for the probability distribution for the position after n steps and to show the scaling of $P_n(x)$ is anomalous. We should begin by finding an expression for the structure function,

$$\hat{p}(k) = \int_{-\infty}^{\infty} e^{-ikx} \frac{A}{(1+x^2)}.$$

This integral is very similar to the integral computed in the last problem, but with simple rather than second order poles at $\pm i$. Thus the same analysis as in the previous problem can be applied. Again the integral is even, and again a contour on the lower half plane, from $-R$ to R on the real axis, will be chosen for the integration, with the restriction $k > 0$. We will define

$$f(z) = e^{-ikz} \frac{A}{(z+i)(z-i)},$$

and use the residue of $f(z)$ at $-i$ to compute the integral,

$$\int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz \underset{R\to\infty}{=} \int_{-R}^{R} f(z)dz = -2\pi i \mathrm{Res}(-i).$$

Using the same arguments as before, the maximum bound for $f(z)$ on C_R is

$$|f(z)| = \left|e^{-ikz}\right| \left|\frac{A}{(1+x^2)}\right| = \left|e^{-ikR(\cos\theta + i\sin\theta)}\right| \left|\frac{A}{(1+R^2)}\right| = \left|e^{kR\sin\theta}\right| \left|\frac{A}{(1+R^2)}\right| = \left|\frac{A}{(1+R^2)}\right|.$$

Accordingly the upper bound for the integral over C_R is

$$\left|\int_{C_R} f(z)dz\right| = \max|f(z)|\, L(C_R) = \frac{\pi R A}{(1+R^2)},$$

which tends to zero as $R \to \infty$. The Residue for a simple pole, $z = a$, can be computed by dividing the numerator of $f(z)$ by the derivative of the denominator, evaluated at a. Thus,

$$Res(-i) = \left[\frac{N\left(f(z)\right)}{D'\left(f(z)\right)}\right]_{z=-i} = \left[\frac{Ae^{-ikz}}{2z}\right]_{z=-i} = \frac{Ae^{-k}}{-2i},$$

and the integral, $\int\limits_{-\infty}^{\infty} f(z)dz$ can now be computed:

$$\hat{p}(k) = \int\limits_{-\infty}^{\infty} f(z)dz = -2\pi i Res(-i) = \pi A e^{-k} \qquad k > 0 \tag{46}$$

As in the previous problem, because $\hat{p}(k)$ is even, $\hat{p}(k) = \hat{p}(-k)$ and to achieve proper normalization of the pdf, $A = \frac{1}{\pi}$, so the structure function is given by

$$\hat{p}(k) = e^{-|k|}. \tag{47}$$

The Taylor expansion of $\hat{p}(k)$ is then given by

$$\hat{p}(k) = e^{-|k|} = 1 - |k| + \frac{|k|^2}{2!} - \frac{|k|^3}{3!} + ... \tag{48}$$

When an expansion of $\log \hat{p}(k)$ is carried out, using (48) one finds

$$\log \hat{p}(k) = -|k| - \frac{2|k|^3}{3} + O\left(k^4\right),$$

which shows the unusual result that the first moment is divergent, due to the absolute value sign, and in fact there is no k^2 term, and thus the Central Limit Theorem does not hold in this case.

The probability distribution, $P_n(x)$, though, can still be found. Proceed by writing the familiar expressions of $\hat{P}_n(k)$ in terms of $\hat{p}(k)$:

$$\log \hat{P}_n(k) = n \log \hat{p}(k) = -n|k| - \frac{2n|k|^3}{3} + O\left(nk^4\right)$$

and

$$\hat{P}_n(k) = e^{n \log \hat{p}(k)} = e^{-n|k| - \frac{2n|k|^3}{3} + O\left(k^4\right)}.$$

Then, $P_n(x)$ is expressed as the inverse Fourier transform,

$$P_n(X) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{ikx} e^{-n|k|} e^{-\frac{2n|k|^3}{3} + O\left(k^4\right)} dk. \tag{49}$$

After expanding the third exponential, as usual, and recognizing that the expansion tends to one, since , as in previous cases, higher order terms in k decrease in negative powers of n, (49) can be expressed as

$$P_n(X) \cong \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{ikx} e^{-n|k|} dk.$$

The steps leading to the evaluation of this integral are:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-n|k|}dk = \frac{1}{2\pi}\int_{-\infty}^{0} e^{ikx}e^{nk}dk + \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx}e^{-nk}dk$$
$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{k(n+ix)}dk + \int_{0}^{\infty} e^{-k(n+ix)}dk\right] = \frac{1}{2\pi}\left[\frac{1}{n+ix}e^{k(n+ix)}\Big|_{-\infty}^{0} - \frac{1}{n-ix}e^{-k(n+ix)}\Big|_{0}^{\infty}\right].$$
$$= \frac{1}{2\pi}\left[\frac{1}{n+ix} - \frac{1}{n-ix}\right] = \frac{n}{\pi(n^2+x^2)}$$

Thus the final result for the probability distribution after n steps is

$$P_n(X) = \frac{n}{\pi\left(n^2+x^2\right)}. \tag{50}$$

By inspection of (50), it can be seen that height of the distribution decreases as $1/n$, since

$$P_{n\max}(X) = \frac{n}{\pi\left(n^2\right)} = \frac{1}{\pi n},$$

and thus, to conserve the zeroth moment of the distribution, the width must increase as n. This truly is anomalous, "super-", diffusion in contrast to the usual $\sqrt{n}$ scaling.

An alternative way to see the scaling of the width is to define a characteristic width, $X_w(n)$, that bounds some arbitrary portion of the pdf, I, such that $I \equiv \int_0^{X_w} P_n(x)dx$, where I is constant with n. Thus I can be determined by,

$$I = \frac{n}{\pi}\int_0^{X_w}\left(n^2+x^2\right)^{-1}dx = \frac{n}{\pi}\left(\frac{1}{n}\right)\tan^{-1}\left(\frac{x}{n}\right)\Big|_0^{X_w} = \frac{1}{\pi}\tan^{-1}\left(\frac{X_w}{n}\right),$$

and we arrive at the scaling relation for $X_w = n\tan(\pi I)$, which, since $\tan(\pi I)$ is a constant, shows that the width scales as n.