Soltuions to Problem Set 1a

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1 Kramers-Moyal Expansion

The solution given here is quite similar to the derivation in H. Risken's book, *The Fokker-Planck Equation*, (Springer, 1989). First define the characteristic function of the transition probability density $p(x, t+\tau|x', t)$ as

$$\hat{p}(k, t+\tau|x', t) = \int dx e^{ik(x-x')} p(x, t+\tau|x', t). \tag{1.1}$$

Suppose the moments

$$M_n(x, t, \tau) = \int p(x+y, t+\tau|x, t) y^n dy \tag{1.2}$$

are finite, the charateristic function $\hat{p}(t+\tau|x', t)$ can be expanded explicitly as

$$\hat{p}(k, t+\tau|x', t) = 1 + \sum_{n=1}^{\infty} \frac{(ik)^n M_n(x', t, \tau)}{n!}. \tag{1.3}$$

The transition probability density $p(x, t+\tau|x', t)$ can be written as inverse Fourier transform of the charateristic function $\hat{p}(t+\tau|x', t)$,

$$\begin{aligned} p(x, t+\tau|x', t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik(x-x')} \hat{p}(t+\tau|x', t) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik(x-x')} [1 + \sum_{n=1}^{\infty} \frac{(ik)^n M_n(x', t, \tau)}{n!}]. \end{aligned} \tag{1.4}$$

Using the ansatz that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (ik)^n e^{ik(x-x')} dk = (-\frac{\partial}{\partial x})^n \delta(x-x'), \tag{1.5}$$

and

$$\delta(x-x') f(x') = \delta(x-x') f(x), \tag{1.6}$$

we obtain

$$p(x,t+\tau|x',t) = [1+\sum_{n=1}^{\infty}(-\frac{\partial}{\partial x})^n \frac{M_n(x',t,\tau)}{n!}]\delta(x-x'). \tag{1.7}$$

Replace the transition probability in Bachelier's equation

$$c(x,t+\tau|x',t) = \int p(x,t+\tau|x',t)c(x',t)dx' \tag{1.8}$$

leads to

$$c(x,t+\tau|x',t) = c(x,t) + \sum_{n=1}^{\infty}(-\frac{\partial}{\partial x})^n[\frac{M_n(x',t,\tau)}{n!}c(x,t)]. \tag{1.9}$$

Taypor expansion of the analytic function $c(x,t+\tau)$ and rearrangement of the above equation give the Kramers-Moyal forward expansion,

$$\frac{\partial c}{\partial t} + \sum_{n=2}^{\infty}\frac{\tau^{n-1}}{n!}\frac{\partial^n c}{\partial t^n} = \sum_{n=1}^{\infty}(-\frac{\partial}{\partial x})^n[D_n(x,t,\tau)c(x,t)] \tag{1.10}$$

where $D_n(x,t,\tau) = M_n(x,t,\tau)/(n!\cdot\tau)$. Assume $D_n(x,t,\tau) \to D_n(x,t)$ as $\tau \to 0$ for $n \le 2$, the Kramer-Moyal expansion is simplified as

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}[D_1(x,t)c(x,t)] + \frac{\partial^2}{\partial x^2}[D_2(x,t)c(x,t)], \tag{1.11}$$

which is so-called Fokker-Planck equation.

2 Diffusion Limited Electrodeposition

The Diffusion equation equation and its boundary condition read

$$\begin{aligned}
\frac{\partial c}{\partial t} &= D\frac{\partial^2 c}{\partial x^2} && (2.1)\\
c(x,0) &= c_0 && (2.2)\\
-D\frac{\partial c}{\partial x}(0,t) &= J && (2.3)\\
c(\infty,t) &= c_0, && (2.4)
\end{aligned}$$

which can be Laplace transformed into

$$\begin{aligned}
s\tilde{c}(x,s) - c_0 &= D\frac{\partial^2 \tilde{c}(x,s)}{\partial x^2} && (2.5)\\
-D\frac{\partial \tilde{c}}{\partial x}(x,s) &= \frac{J}{s} && (2.6)\\
\tilde{c}(x,s) &= \frac{c_0}{s}. && (2.7)
\end{aligned}$$

The differential equation (2.5) with the boundary equation (2.7) has the solution

$$\tilde{c}(x,s) = \tilde{B}(s)e^{-\sqrt{\frac{s}{D}}x} + \frac{c_0}{s}, \tag{2.8}$$

where $\tilde{B}(s)$ is a function of s to be defined. Combined with the Eq. (2.6), we obtain

$$-D\tilde{B}(s)(-\sqrt{\frac{s}{D}}) = \frac{J}{s}, \tag{2.9}$$

thus leads to

$$\tilde{B}(s) = \frac{J}{\sqrt{D}} s^{-\frac{3}{2}}. \tag{2.10}$$

So the complete solution in the Laplace space is

$$\tilde{c}(x,s) = \frac{J}{\sqrt{D}} s^{-\frac{3}{2}} e^{-\sqrt{\frac{s}{D}}x} + \frac{c_0}{s}. \tag{2.11}$$

Explicit inverse Laplace transform gives the $c(x,t)$ as

$$c(x,t) = c_0 + \frac{J}{D}[2(\frac{Dt}{\pi})^{\frac{1}{2}} e^{-\frac{x^2}{4Dt}} - x\mathrm{Erfc}(\frac{x}{2\sqrt{Dt}})]. \tag{2.12}$$

Erfc(x) is the complementary error function defined as

$$\mathrm{Erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \tag{2.13}$$

The concentration at $x = 0$ can be derived directly from Eq. (2.5),

$$\tilde{c}(0,s) = \frac{J}{\sqrt{D}} s^{-\frac{3}{2}} + \frac{c_0}{s}, \tag{2.14}$$

therefore,

$$c(0,t) = \frac{J}{\sqrt{D}} \frac{2t^{\frac{1}{2}}}{\sqrt{\pi}} + c_0. \tag{2.15}$$

In the case that $J < 0$, the concentration at $x = 0$ eventually vanishes at the Sand's time t_s,

$$c(0,t_s) = \frac{J}{\sqrt{D}} \frac{2t_s^{\frac{1}{2}}}{\sqrt{\pi}} + c_0 = 0, \tag{2.16}$$

therefore leads to

$$t_s = \frac{D\pi c_0^2}{4J^2}. \tag{2.17}$$

After the Sand's time t_s, the current J can no longer be maintained due to the diffusion limitation.