# Financial Derivatives

18.095: IAP Mathematics Lecture Series 2000
**Solutions** to Homework Problems for Lecture 5

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1. If $c(x,t,K)$ and $p(x,t,K)$ are the fair prices for call and put options struck at $K$, respectively, then the fair price for the butterly spead $w(x,t,a,b)$ is

$$\begin{aligned} w(x,t,a,b) &= c(a,t,a-b) - 2c(x,t,a) + c(x,t,a+b), \text{ or} && (1) \\ &= p(a,t,a-b) - 2p(x,t,a) + p(x,t,a+b) && (2) \end{aligned}$$

which holds for any $(x,t)$, not only at maturity. In other words, being long on the butterfly spread is equivalent to being long on calls (or puts) struck at $a-b$ and $a+b$ and short on two calls (or puts) struck at $a$.

2. The initial option value is $w = u_o + \phi x_o$, which corresponds to a replicating portfolio of $u_o$ in cash and $\phi x_o$ in the underlying. At maturity, this portfolio has value $u_o + \phi x$, which must equal the payoff $y = ax + b$ (ignoring interest-rate effects). Therefore, a riskless hedge (for any random variable $x$) is possible if we choose $u_o = b$ and $\phi = a$, which yields the unique price

$$w(x_o) = ax_o + b. \tag{3}$$

Note that the Bachelier price $\langle y \rangle = a\langle x \rangle + b$ is fair if and only if the expected return vanishes, $x_o = \langle x \rangle$. Plugging $y = ax + b$ into the quadratic risk formula yields the same result

$$\phi^* = \frac{\langle xy \rangle - \langle x \rangle\langle y \rangle}{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a\sigma_x^2}{\sigma_x^2} = a. \tag{4}$$

3. For the binomial model, we can show explicitly that $\sigma_x^2 = p_1 p_2 (x_2 - x_1)^2$, $R_B^2 = \sigma_y^2 = p_1 p_2 (y_2 - y_1)^2$, $\phi^* = (y_2 - y_1)/(x_2 - x_1)$ and hence $R^{*2} = \sigma_y^2 - \phi^{*2}\sigma_x^2 = 0$.

The "risk-neutral" (binomial) probability distribution is determined from $\langle x \rangle = p_1^* x + (1 - p_1^*)x_2 = x_o$, which implies $p_1^* = (x_2 - x_o)/(x_2 - x_1)$ and $p_2^* = 1 - p_1^* = (x_o - x_1)/(x_2 - x_1)$. Observe that the risk-neutral valuation formula $w = \langle y \rangle^* = p_1^* y_1 + p_2^* y_2$ is the same as the (linear interpolation) formula for the fair price derived in the lecture.

4. Assume the following uniform transition probability

$$p(x|x_o) = \begin{cases} \frac{1}{2a} & \text{if } x_o - a \le x \le x_o + a \\ 0 & \text{otherwise} \end{cases}$$

for the underlying asset. Note that if $x_o \le K - a$, then the payoff is linear ($y = 0$) for $x \in (x_o - a, x_o + a)$, and hence the unique fair price is $w(x_o) = 0$, according to problem 2 above.

Similarly, if $x_o \geq K + a$, the unique fair price is $w(x_o) = x_o - K$, as for a forward contract. However, if $K - a < x_o < K + a$, then the price must be determined by risk minimization, because the option contract has a nonzero chance of reaching the positive payoff $x > K$ but at the same time has a nonzero chance of not being exercise $x < K$.

Note that $\langle x \rangle = x_o$, and therefore, the Bachelier formula gives the "fair price" which minimizes the Bouchaud-Sornette quadratic risk:

$$w(x_o) = \langle y \rangle = \int_K^{x_o+a} \frac{x-K}{2a} dx. \tag{5}$$

Performing this trivial integral, we find

$$w(x_o) = \begin{cases} 0 & \text{if} x_o \leq K - a \\ \frac{(x_o+a-K)^2}{4a} & \text{if} K - a < x_o < K + a \\ x_o - K & \text{if} x_o \geq K + a \end{cases} \tag{6}$$

Try plotting this function to see that it is continuous. Note how it looks a bit like the solution to the Black-Scholes equation for a call option which was sketched in the lecture.

The absolute-value risk for $K - a < x_o < K + a$ is given by

$$R'(x_o) = \int_{x_o-a}^{x_o+a} \frac{|y(x) - (u_o + \phi x)|}{2a} dx \tag{7}$$

where $y(x) = \max(x - K, 0)$. Therefore, $2aR'(x_o)$ is the sum of the (non-negative) areas between the curves $y(x)$ and $u_o + \phi x$. By symmetry, the minimum risk $R'^*(x_o)$ is attained when there are equal areas above and below the line $u_o^* + \phi^* x$. The optimal line $w = u_o^* + \phi^* x$ is defined by two points $x_1$ and $x_2$, where it intersects the $y = 0$ and $y = x - K$ pieces of the payoff function, respectively:

$$w(x) = \left(\frac{x_2 - K}{x_2 - x_1}\right)(x - x_1). \tag{8}$$

From the "equal-area argument" we have

$$x_1 = \frac{1}{2}(x_o - a + K) \tag{9}$$

$$x_2 = \frac{1}{2}(x_o + a + K) \tag{10}$$

which are the two the midpoints of the segments fo the zero and positive payoffs within the range of the uniform transition probability. To see this geometrically, try drawing a picture! Substituting (9) and (10) into (8) we obtain the *same* result for the fair price $w(x_o)$ as in (6). However, note that the value of residual risk (and hence the market cost of the option) depends on the choice of risk function. (Try computing $R'^*$ and $R^*$.)