Möbius, Chebychev and Césaro on Series Inversion

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January 13, 1999

Abstract

A historical account is presented of the problem of inverting series of the form, $\sum_{n=1}^{\infty} a(n) f(\varepsilon_n(x))$, which has led to important developments in number theory, combinatorics and, more recently, physics.

1 The Pioneering Paper of Möbius

The broad topic of "Möbius inversion" in number theory [1, 2, 3, 4] and combinatorics [5] can be traced to an 1832 paper by August Ferdinand Möbius (1790–1868). We begin with a translation from the original German of this paper [6] (which is also reproduced in his collected works [7]).

On One Special Type of Inversion of Series.
(by Mr. *A. F. Möbius*, Professor in Leipzig.)

The famous problem of inversion of a series is known to consist in that, when a function of a variable is defined by a series of increasing powers, one requires the inversion of the variable itself, or any other function of it, to be expressed through a series of progressing powers of that function. As is well known, no small analytical skills are needed to find the law that relates the coefficients of the second series to those of the first one. Incomparably simpler is the following problem.

Let fx be a function of a variable x given as an ordered series of powers of x:

$$1. \quad fx = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots.$$

The variable x is to be expressed not by means of powers of fx but by means of functions f of the powers of x, as represented by the following series:

$$2. \quad x = b_1 fx + b_2 f(x^2) + b_3 f(x^3) + b_4 f(x^4) + \dots.$$

Whereas this problem can be posed either in the context of the difficulties of its solution or in the context of its uses in the aforementioned problem of series inversion, in which neither the values of fx, $f(x^2)$, $f(x^3)$,... nor the value of x can be calculated through formula (2.), thus leaving real purpose of the inversion problem unattained, the solution of this problem will nevertheless lead to many results not unimportant for the theory of series as well as the theory of combinatorics.

The essence of our problem is this: to express the coefficients b_1, b_2, b_3, ... of series (2.) as functions of the coefficients a_1, a_2, a_3, ...; and this can be done through the following very easy calculations. From (1.) it follows:

$$\begin{array}{lcl}
f(x^2) & = & a_1x^2 + a_2x^4 + a_3x^6 + a_4x^8 + \ldots \\
f(x^3) & = & a_1x^3 + a_2x^6 + a_3x^9 + \ldots \\
f(x^4) & = & a_1x^4 + a_2x^8 + \ldots \\
f(x^5) & = & a_1x^5 + a_2x^{10} + \ldots \\
f(x^6) & = & a_1x^6 + a_2x^{12} + \ldots \\
\text{et cetera} & &
\end{array}$$

Substituting these values of $f(x^2)$, $f(x^3)$, ... and of fx directly into equation (2.), one obtains:

$$x = a_1b_1x \begin{array}{l|} +a_2b_1 \\ +a_1b_2 \end{array}\, x^2 \begin{array}{l|} +a_3b_1 \\ +a_1b_3 \end{array}\, x^3 \begin{array}{l|} +a_4b_1 \\ +a_2b_2 \\ +a_1b_4 \end{array}\, x^4 \begin{array}{l|} +a_5b_1 \\ +a_1b_5 \end{array}\, x^5 \begin{array}{l|} +a_6b_1 \\ +a_3b_2 \\ +a_2b_3 \\ +a_1b_6 \end{array}\, x^2 + \ldots.$$

The rule of progression of the coefficients of this series is obvious. If just the coefficient of x^m is to be determined, one needs to decompose the number m in all possible ways into two positive factors. Each of these products then gives one term of the coefficient sought, as a result of which one takes two factors of the product as the indices of the mutually multiplied a and b.

Since the equation just obtained must apply for each value of x, we have

$$3. \quad \begin{cases}
a_1b_1 = 1, \\
a_2b_1 + a_1b_2 = 0, \\
a_3b_1 + a_1b_3 = 0, \\
a_4b_1 + a_2b_2 + a_1b_4 = 0, \\
a_5b_1 + a_1b_5 = 0, \\
a_6b_1 + a_3b_2 + a_2b_3 + a_1b_6 = 0, \\
\text{et cetera}
\end{cases}$$

through which each b can be calculated with the help of the preceding b.

To find, in this manner, the individual b independently of each other, one may set for the sake of simplicity $a_1 = 1$, and it ensues

$$4. \quad \begin{cases}
b_1 = 1, \\
b_2 = -a_2, \\
b_3 = -a_3, \\
b_4 = -a_4 + a_2a_2, \\
b_5 = -a_5, \\
b_6 = -a_6 + a_3a_2 + a_2a_3, \\
b_7 = -a_7, \\
b_8 = -a_8 + a_4a_2 + a_2a_4 - a_2a_2a_2, \\
\text{et cetera}
\end{cases}$$

Already these limited developments suffice for inferring how also the values of the successive b must be assembled from a_1, a_2, ... Indeed, one decomposes index m in all possible ways into factors, taking m itself as the highest factor but omitting the unity, and regarding each two decompositions that differ only by the order of their factors as different. Or, as can be concisely stated in the language of combinatorics: One constructs all the variations with repetitions for the product m. Each of these variations thus yields one term in the value of b_m, which takes the index m in the elements of variation, and this term receives either a positive or negative sign, depending on whether the number of its elements is even or odd.

So, for instance, the variations of product 12 are:

$$12,\ 2\cdot 6,\ 3\cdot 4,\ 4\cdot 3,\ 6\cdot 2,\ 2\cdot 2\cdot 3,\ 2\cdot 3\cdot 2,\ 3\cdot 2\cdot 2,$$

and from this

$$b_{12} = -a_{12} + 2a_2a_6 + 2a_3a_4 - 3a_2a_2a_3.$$

The general validity of this law ensues from the recursion formula (3.) so easily that it would be superfluous to occupy ourselves with the proof.

These relations among the coefficients a and b could have also been obtained if one mutually compared, similar to (1.) and (2.), in the same way the general equations

$$\begin{aligned} fx &= a_1Fx + a_2F(x^2) + a_3F(x^3) + \ldots \\ Fx &= b_1fx + b_2f(x^2) + b_3f(x^3) + \ldots. \end{aligned}$$

Instead, one would then obtain between the a and b the relations (3.), and the first as a result of the second, which, in the former case Fx, and in the latter fx, could be taken as a function of x.

Relations (3.) hold true, as one easily verifies, also when one substitutes

$$\begin{array}{llll} \text{for} & a_2, a_3, a_4, \ldots & \text{respectively} & 2^na_2, 3^na_3, 4^na_4, \ldots \text{ and} \\ \text{for} & b_2, b_3, b_4, \ldots & \text{respectively} & 2^nb_2, 3^nb_3, 4^nb_4, \ldots \end{array}$$

where n can be an arbitrary positive or negative whole number or a fraction. When at the same time

$$A.\quad \begin{cases} fx = a_1Fx + 2^na_2F(x^2) + 3^na_3F(x^3) + \ldots, \text{then also} \\ Fx = b_1fx + 2^nb_2f(x^2) + 3^nb_3f(x^3) + \ldots, \end{cases}$$

and vice versa.

Of even greater generality then these, are the following two companion equations:

$$B.\quad \begin{cases} fx = a_1Fx + c_2a_2F(x^2) + c_3a_3F(x^3) + \ldots, \\ Fx = b_1fx + c_2b_2f(x^2) + c_3b_3f(x^3) + \ldots. \end{cases}$$

In them, the coefficients c_2, c_3, $\ldots$, whose indices are primes, can be set as desired. One coefficient c, which is a composite number, must then be taken equal to the product of the coefficients c whose indices are the simple factors of that composite number; also $c_4 = c_2^2$, $c_6 = c_2 \cdot c_3$, etc. The required relations between the a and the b are the same as before.

To give a very simple example of this new kind of series inversion, we may set

$$a_1 = a_2 = a_3 = \ldots = 1,$$

also from(1.)

$$fx = x + x^2 + x^3 + \ldots \text{ and from that } fx = \frac{x}{1-x}.$$

With these values of a, however, according to (4.):

$$b_1 = 1,\ b_2 = -1,\ b_3 = -1,\ b_4 = 0,\ b_5 = -1,\ b_6 = 1,\ b_7 = -1,\ b_8 = 0,$$

etc., and then from (2.):

$$[1.] \quad x = \frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} - \frac{x^7}{1-x^7} + \frac{x^{10}}{1-x^{10}} - \frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}} + \text{ etc.}$$

One would surely find it very pleasing to find that the coefficients of this series, when it is continued ever farther, are none other than 1, 0 and -1. The basis of this remarkable result, and the law according to which the coefficients 1, 0 and -1 interchange with each other, can be most easily uncovered with the help of recursion formula (3.)

At this point, Möbius goes on to explore the properties of the "remarkable" coefficients b_n, which, of course, form the familiar Möbius function $\mu(n)$. He argues that $b_n = 0$ for numbers divisible by a square, and that the coefficients alternate in sign based on the number of distinct prime factors. Using the modern notation $\mu(n)$ for b_n, he derives the formula:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisble by a square} \end{cases} \tag{1}$$

By considering a succession of special cases, he also derives the divisor sum rule,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1, \end{cases} \tag{2}$$

which is the essential property of the Möbius function.

Möbius concludes the paper by deriving a number of fascinating series expansions, similar to [1.] (which incidentally was derived in a different way by Laguerre [11]). Möbius' clever idea is to work with functions whose Taylor expansion coefficients are multiplicative, and hence easily invertable using his formulae. For example, since $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$ for $|x| < 1$, formula (A.) implies (with $n = -1$),

$$[13.] \quad e^x = (1-x)^{-1}(1-x^2)^{\frac{1}{2}}(1-x^3)^{\frac{1}{3}}(1-x^5)^{\frac{1}{5}}(1-x^6)^{-\frac{1}{6}} \text{ etc..}$$

Note that this does not establish a product expansion for e, since $x = 1$ is just at the radius of convergence, but it does, for example, yield a curious relationship between $\sqrt{e}$ and $\mu(n)$, namely $\sqrt{e} = \prod(1-2^{-n})^{-\mu(n)/n}$. Using a related formula to invert the Taylor series, $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$, and evaluating at $x = 1$, Möbius derives an expansion for $1/\pi$, again involving $\mu(n)$,

$$[24.] \quad \frac{4}{\pi} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} - \frac{1}{17} + \text{ etc.,}$$

where the coefficient of $\frac{1}{2k+1}$ is $(-1)^k\mu(2k+1)$. This method of deriving series formulae by inversion of Taylor (or other kinds of) expansions, pioneered by Möbius, is still in use today [2, 22].

In terms of the Dirichlet inverse $a^{-1}(n)$, the results of Möbius are best summarized with the following formula:

$$F(x) = \sum_{n=1}^{\infty} a(n) f(x^n) \iff f(x) = \sum_{n=1}^{\infty} a^{-1}(n) F(x^n). \tag{3}$$

When considering the case $a(n) = 1$ for all n leading to Eq. [1.], Möbius is led to the function bearing his name as the inversion coefficients. In current notation, therefore, Möbius has actually defined $\mu(n)$ via $\mu = u^{-1}$, where $u(n) = 1$ for all n, and it is from this relation that the properties of Eq. (1) are deduced.

In general, Dirichlet inverses must be calculated recursively using 4., but there is a special class of functions for which the inverse is easy to write down, namely those that are *completely multiplicative* [2], $f(mn) = f(m)f(n)$ for all m and n. If f is completely multiplicative, then $f^{-1}(n) = f(n)\mu(n)$ for all n. The appearance of $\mu(n)$ in this formula is the essential reason for its importance in number theory. Although to the author's knowledge this result has not been attributed to Möbius, he has clearly derived it (with greater generality) in formula (B.), which states that, if c is completely multiplicative, the Dirichlet inverse of $c(n)a(n)$ is $c(n)a^{-1}(n)$. Similiar results were later derived by others [3], notably Cesàro [18, 19], but Möbius was apparently the originator.

Incidentally, according to Dickson [3] it was Mertens who introduced the notation $\mu(n)$. In a later textbook on number theory, P. Bachmann actually refers to it as the "Mertens function", without reference to Möbius [20]. Since Mertens does not cite Möbius either [14, 15], it appears that he did not choose the symbol μ to be identified with the name "Möbius", although coincidentally "Mertens" begins with the same letter.

2 Independent Invention by Chebychev

Various nineteenth century authors studying series inversion [14, 18, 19] take inspiration from an 1851 paper by P. L. Chebychev [13]. This important paper is not cited anywhere in Dickson's historical treatise [3]. It is worth rediscovering Chebychev's paper because he derives some of the results of Möbius using more elegant methods that influenced later authors. Thus, we now present an unabridged translation from French of the opening section of Chebychev's paper [13] (the original version was written in Russian):

NOTE ON VARIOUS SERIES;

by Mr. P. CHEBYCHEV.

One arrives at some very interesting formulae, such as, for example, the expansion of e^{-x} in a product of an infinity of factors, the expression for $\sin x$ and $\cos x$ in series whose calculation requires only the multiplication of x by various constants, and also in several other curious relations, in seeking the value of $f(1)$, $f(x)$ being a function tied to another $F(x)$ by an equation of one of these three forms:

$$\begin{aligned}
F(x) &= f(x) + f(2x) + f(3x) + f(4x) + f(5x) + f(6x) + \dots, \\
F(x) &= f(x) + f(3x) + f(5x) + f(7x) + f(9x) + f(11x) + \dots, \\
F(x) &= f(x) - f(3x) + f(5x) - f(7x) + f(9x) - f(11x) + \dots.
\end{aligned}$$

To succeed in this, we are going to seek what is the law of these series which determines the value of $f(1)$ in these three cases. We begin with the first where $f(x)$ and $F(x)$ are tied by the equation

$$(1) \quad Fx = f(x) + f(2x) + f(3x) + f(4x) + f(5x) + f(6x) + \dots.$$

It is clear that by virtue of this equation, the desired value of $f(1)$ will be expressed by a series of the following form:

$$(2) \quad f(1) = A_1F(1) + A_2F(2) + A_3F(3) + \ldots + A_mF(m) + \ldots,$$

where $A_1, A_2, A_3, \ldots, A_m$ are some numerical coefficients independent of the functions $f(x)$ and $F(x)$. In order to find these coefficients, we assume

$$f(x) = \frac{1}{x^r},$$

r being some quantity greater than 1. For this particular form of $f(x)$, the function $F(x)$, according to equation (1), will be determined by the equation

$$F(x) = \frac{1}{x^r} + \frac{1}{(2x)^r} + \frac{1}{(3x)^r} + \frac{1}{(4x)^r} + \frac{1}{(5x)^r} + \frac{1}{(6x)^r} + \ldots,$$

or just as well, what ammounts to the same thing,

$$F(x) = \frac{1}{x^r} \cdot \frac{1}{1 - \frac{1}{2^r}} \cdot \frac{1}{1 - \frac{1}{3^r}} \cdot \frac{1}{1 - \frac{1}{5^r}} \cdot \frac{1}{1 - \frac{1}{7^r}} \cdot \cdots.$$

Therefore, since this expression for $F(x)$, together with

$$f(x) = \frac{1}{x^r},$$

must satisfy equation (2), we conclude that

$$1 = \frac{1}{1 - \frac{1}{2^r}} \cdot \frac{1}{1 - \frac{1}{3^r}} \cdot \frac{1}{1 - \frac{1}{5^r}} \cdot \frac{1}{1 - \frac{1}{7^r}} \cdots \left(A_1 + \frac{A_2}{2^r} + \frac{A_3}{3^r} + \ldots + \frac{A_m}{m^r} + \ldots \right),$$

and, consequently,

$$A_1 + \frac{A_2}{2^r} + \frac{A_3}{3^r} + \ldots + \frac{A_m}{m^r} + \ldots = \left(1 - \frac{1}{2^r}\right)\left(1 - \frac{1}{3^r}\right)\left(1 - \frac{1}{5^r}\right)\left(1 - \frac{1}{7^r}\right) \cdots.$$

From this equation, and in remarking that the coefficients $A_1, A_2, A_3, \ldots, A_m, \ldots$ are independent of r, one concludes that in general A_m is equal to the coefficient of $\frac{1}{m^r}$ in the expansion of the product

$$\left(1 - \frac{1}{2^r}\right)\left(1 - \frac{1}{3^r}\right)\left(1 - \frac{1}{5^r}\right)\left(1 - \frac{1}{7^r}\right) \cdots.$$

Now, in examining this product, one perceives in succession that there do not appear any terms of the form $\frac{1}{m^r}$ whenever m is divisible by a square; therefore, for these values of m, the coefficient A_m will be equal to zero. As for the other values of m, the coefficient A_m reduces to 1 whenever m has an even number of prime divisors, and it is equal to -1 when the number of prime divisors of m is even. Thus we find that the law of the series (2) which gives the value of $f(1)$ is determined by the equation [(1)].

Chebychev goes on to make similar arguments for the other two cases. His results can be expressed in more familiar notation as follows [18]:

$$F(x) = \sum_{n=1}^{\infty} f(nx) \quad \Longleftrightarrow \quad f(x) = \sum_{n=1}^{\infty} \mu(n)F(nx), \tag{4}$$

$$F(x) = \sum_{k=1}^{\infty} f((2k-1)x) \quad \Longleftrightarrow \quad f(x) = \sum_{k=1}^{\infty} \mu(2k-1)F((2k-1)x) \tag{5}$$

$$F(x) = \sum_{k=1}^{\infty} (-1)^k f((2k-1)x), \quad \Longleftrightarrow \quad f(x) = \sum_{k=1}^{\infty} (-1)^k \mu(2k-1)F((2k-1)x). \tag{6}$$

Using these formulae, Chebychev derives results analogous to [1.], [13.] and [24.] of Möbius. He also considers inversion of Fourier series. For example, starting with the Fourier expansion of a triangle wave,

$$\frac{\pi^2}{8}(1-4\{x\}) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos[2\pi(2k-1)x],$$

where $\{x\}$ denotes the closest integer to x, he derives an expansion for $\cos(2\pi x)$ in terms of the triangle wave,

$$\cos(2\pi x) = 1 - \frac{\pi^2}{2}\sum_{k=1}^{\infty}\frac{\mu(2k-1)}{(2k-1)^2}\{(2k-1)x\},$$

using Eq. (5). With similar methods, he also generates quickly convergent expansions for sums of the form,

$$\Sigma_m = \frac{(-1)^m}{3^m} + \frac{1}{5^m} + \frac{(-1)^m}{7^m} + \frac{(-1)^m}{11^m} + \frac{1}{13^m} + \ldots = \sum_{k=1}^{\infty}\Omega(2k+1)\frac{(-1)^{km}}{(2k+1)^m},$$

where $\Omega(n) = 1$ if n is a prime and vanishes otherwise. Although the series of inverse primes diverges was known to diverge since the time of Euler, Chebychev argues that this alternating series of inverse primes converges to $\Sigma_1 = -0.33498\ldots$, and also concludes that $\Sigma_2 = 0.20224\ldots$ and $\Sigma_3 = -0.03225\ldots$.

3 Synthesis by Césaro

The study of the Möbius-type series inversion formulae flourished in the latter half of the nineteenth century. Following Möbius and Chebychev, many related inversion formulae were derived by E. Laguerre[11], E. Meissel[12], F. Mertens[14, 15], J. W. L. Glaisher[16], E. Cesàro[18, 19], P. Bachmann[20] and E. Landau[21]. The authors also make reference to related work by J. J. Sylvester, E. Catalan, J. Liouville, G. L. Dirichlet, H. J. S. Smith and many others. Typical uses of the inversion formulae include derivations of series or product expansions for irrational numbers like π, analytic functions like $\cos(x)$ or number-theoretic functions like $\phi(n)$. It is important to note that most of these early studies were not rigorous, and more careful treatments of some of the same problems have appeared in this century [22].

It is natural to ask what is the most general inversion formula that contains the aforementioned ones as special cases. This question was answered by E. Cesàro in 1885 [18]. Ignoring issues of convergence, he derives the following theorem: If $\varepsilon_m(\varepsilon_n(x)) = \varepsilon_{mn}(x)$ for all whole numbers m and n and real numbers x, then,

$$F(x) = \sum_{n=1}^{\infty} a(n) f(\varepsilon_n(x)) \iff f(x) = \sum_{n=1}^{\infty} a^{-1}(n) F(\varepsilon_n(x)). \tag{7}$$

He refers to a and a^{-1} as "conjugutes", rather than Dirichlet inverses.

In 1887, Cesàro pointed out that the summation in his first theorem need not extend over all whole numbers [19], which Möbius also understood (*e. g.* Eq. [24.]) as did Chebychev (*e. g.* Eq. (5)). Instead, $a(n)$ and $a^{-1}(n)$ can be multiplied by an "enumerative function" $\Omega(n)$, which is 1 for certain integers and 0 for others with the restriction that $\Omega(mn) = \Omega(m)\Omega(n)$. This possiblity, however, is already contained in Cesàro's first theorem since $\Omega(n)$ is completely multiplicative. (Cesàro refers to the set $\{\Omega(n)\}_{n=1}^{\infty}$ as a "group", but in the modern sense it is only a semigroup, since the inverses $\{\Omega(1/n)\}_{n=1}^{\infty}$ are not included in the set.) In 1884, Glaisher also gave a somewhat more general statement of the Möbius formula, Eq. (3), where summation is restricted to certain subsets of the integers[16], but it too is just another special case of Cesàro's theorem. Cesàro cites Glaisher in his 1887 paper[19], so perhaps Glaisher was an important influence along with Chebychev and Möbius (who are also cited).

The sets (or rather sequences) of functions $\{\varepsilon_n(x)\}$ defined by Cesàro clearly play an important role in series inversion. Each sequence has the structure of a multiplicative semigroup under composition, $\varepsilon_m(\varepsilon_n(x)) = \varepsilon_{mn}(x)$, so it would be natural to call such a set of functions a *Cesàro semigroup*. At first it might seem that the only types of Cesàro semigroups are the ones generalizing the formulae (3) and (4) of Möbius and Chebychev, respectively: $\{x^{b(n)}\}$ and $\{b(n)x\}$, where $b(n)$ is completely multiplicative. Indeed, Cesàro mentions these cases, but he also gives several more which are not immediately obvious [19],

$$\varepsilon_n(x) = \frac{x}{1 - x\log b(n)} \quad \text{and} \quad \varepsilon_n(x) = \frac{xb(n)}{1 - kx[b(n) - 1]}, \tag{8}$$

where $b(n)$ again is completely multiplicative and k is a constant. Let us note another example, $\varepsilon_n(x) = x + k\log b(n)$. It would certainly be useful to further enumerate the set of Cesàro semigroups because each one produces a new family of inversion formulae, but to the author's knowledge this has not been done. Therefore, even in such an old field as Möbius series inversion, there may be room for further progress, and the impetus for such mathematical research is provided by the newly discovered possibility of applications in physics [23, 24, 25, 26].

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